Observing quantum criticality at finite temperature through nonanalytic correlation times

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joint work with
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SciPost Phys. **17**, 162 (2024)
arXiv:2602.00794

Qopen COST Action Kick-off meeting
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Outline

- Introduction
- Dynamical correlations in the Ising field theory
- Dynamical correlations in the Ising spin chain

Message: the correlation length has non-analytic dependence on the parameters
(temperature, magnetic field, space-time direction)

Quantum Ising chain

$$H = -J \sum_{j=1}^{N} \left(\sigma_j^x \sigma_{j+1}^x + h \sigma_j^z \right)$$

TFIM, QIC, ...

Quantum critical point: $h_c = 1$

$h > 1$: Disordered (paramagnetic)

$h < 1$: Ordered (ferromagnetic)

in GS: $\langle 0 | \sigma_j^x | 0 \rangle = 0$ $\qquad \langle 0 | \sigma_j^x | 0 \rangle \neq 0$

Jordan-Wigner transformation (nonlocal!)

$$\sigma_j^z = 1 - 2c_j^\dagger c_j, \qquad \sigma_j^x = \prod_{l=0}^{j-1} \left(1 - 2c_l^\dagger c_l \right) (c_j + c_j^\dagger)$$

Bogoliubov rotation in momentum space

$$H = \sum_k \epsilon(k)\, \alpha_k^\dagger \alpha_k + E_0(N) \qquad \epsilon(k) = 2J\sqrt{1 - 2h\cos(k) + h^2}$$

Scaling limit

$$h \to 1,\, a \to 0,\, J \to \infty : \quad Na = L = \text{fixed}, \quad 2J|1-h| = mc^2 = \text{fixed}, \quad 2Ja = c = \text{fixed}$$

$$\varepsilon(p) = \sqrt{m^2c^4 + p^2c^2}$$

Goal: compute dynamical correlation functions of the magnetization at finite temperature

$$C(j, t; \beta, h) = \langle \sigma_j^x(t)\, \sigma_0^x(0) \rangle_\beta = \frac{\mathrm{Tr}\{ e^{-\beta H} \sigma_j^x(t)\, \sigma_0^x(0) \}}{Z}$$

Earlier results (incomplete)

Differential equations for the correlators

J. H. H. Perk, H. W. Capel, G. R. W. Quispel, F. W. Nijhoff, Physica A **123**, 1 (1984)

J. H. H. Perk, H. Au-Yang, J. Stat. Phys. **135**, 599 (2009)

B. M. McCoy, J. H.H. Perk, R. E. Shrock, Nucl. Phys. B **220**, 35 & 269 (1983)

Semiclassical approach at low temperatures

S. Sachdev, A. P. Young, Phys. Rev. Letters **78**, 2220 (1997)

Form factor series

A. LeClair, G. Mussardo, Nucl. Phys. B **552**, 624 (1999)

B. Altshuler, R. Konik, A. Tsvelik NPB **739**, 311 (2006)

B. Doyon, SIGMA **3**, 011 (2007)

E. Granet, M. Fagotti, F. Essler, SciPost Phys. **9**, 033 (2020)

(Quantum) transfer matrix, form factors (mostly on XX and XXZ)

Göhmann, Klümper, Kozlowski, Maillet, Suzuki, Terras, ...

Correlations in the Ising field theory

István Csépányi, M.K., SciPost Phys. **17**, 162 (2024)

Field theory: Finite temperature

$$C(x,t;\beta) = \langle\sigma(x,t)\,\sigma(0,0)\rangle_\beta = m^{1/4}\tilde{C}\left(mx, mt; m\beta\right) \qquad \sigma(x=ja) = (2a)^{-1/8}\sigma_j^z$$

Finite T form factor series for space-like separations: $x^2 - t^2 > 0$

B. Altshuler, R. Konik, A. Tsvelik NPB **739**, 311 (2006); B. Doyon, SIGMA **3**, 011 (2007)

$$C(x,t;\beta) = m^{1/4}S(m\beta)\,e^{-\Delta\mathscr{E}(\beta)x}\sum_{N=0}^{\infty}{}' \frac{1}{N!}\sum_{\epsilon_1,\ldots,\epsilon_N=\pm}\prod_{j=1}^{N}\int_{-\infty+i\epsilon_j\delta}^{\infty+i\epsilon_j\delta}\frac{d\theta_j}{2\pi}\frac{e^{im\epsilon_j(x\sinh\theta_j - t\cosh\theta_j)+\epsilon_j\eta(\theta)}}{\epsilon_j\left(1-e^{-\epsilon_j m\beta\cosh\theta_j}\right)}\prod_{1\le k<l\le N}\tanh\left(\frac{\theta_k-\theta_l}{2}\right)^{2\epsilon_i\epsilon_j}$$

sum over particles in the trace and in the inserted state

Bose-Einstein statistics!

$$\Delta\mathscr{E}(\beta) = \int_{-\infty}^{\infty}\frac{d\theta}{2\pi}m\cosh\theta\log\left[\frac{1+e^{-m\beta\cosh\theta}}{1-e^{-m\beta\cosh\theta}}\right]$$

Energy difference between NS and R ground states on a ring of length β

$$\eta(\theta) = \int_{-\infty}^{\infty}\frac{d\theta'}{\pi i}\frac{1}{\sinh(\theta-\theta')}\log\left[\frac{1+e^{-m\beta\cosh\theta'}}{1-e^{-m\beta\cosh\theta'}}\right]$$

$$S(m\beta) = \exp\left[\frac{(m\beta)^2}{2}\int_{-\infty}^{\infty}\frac{d\theta_1 d\theta_2}{(2\pi)^2}\frac{\sinh\theta_1\sinh\theta_2}{\sinh(m\beta\cosh\theta_1)\sinh(m\beta\cosh\theta_2)}\log\left|\coth\left(\frac{\theta_1-\theta_2}{2}\right)\right|\right]$$

Field theory: Finite temperature

$$C(x,t;\beta) = m^{1/4} S(m\beta)\, e^{-\Delta\mathscr{E}(\beta)x} \sum_{N=0}^{\infty}{}' \frac{1}{N!} \sum_{\epsilon_1,\ldots,\epsilon_N=\pm} \prod_{j=1}^{N} \int_{-\infty+i\epsilon_j\delta}^{\infty+i\epsilon_j\delta} \frac{d\theta_j}{2\pi} \frac{e^{im\epsilon_j(x\sinh\theta_j - t\cosh\theta_j)+\epsilon_j\eta(\theta)}}{\epsilon_j\left(1-e^{-\epsilon_j m\beta\cosh\theta_j}\right)} \prod_{1\le k<l\le N} \tanh\left(\frac{\theta_k-\theta_l}{2}\right)^{2\epsilon_i\epsilon_j}$$

Can be written as a Fredholm determinant:

A. Leclair et al., Nucl. Phys. B **482**, 579 (1996); B. Doyon, SIGMA **3**, 011 (2007)

$$C_\pm(x,t;\beta) \equiv C_\mathrm{f}(x,t;\beta) \pm C_\mathrm{p}(x,t;\beta) = m^{1/4}\, S(m\beta)\, e^{-\Delta\mathscr{E}x}\, \mathrm{Det}\left(\mathbb{I} + \tilde{\mathbf{K}}^{\pm}_{x,t;\beta}\right)$$

The kernel is a 2x2 matrix in ϵ, ϵ'

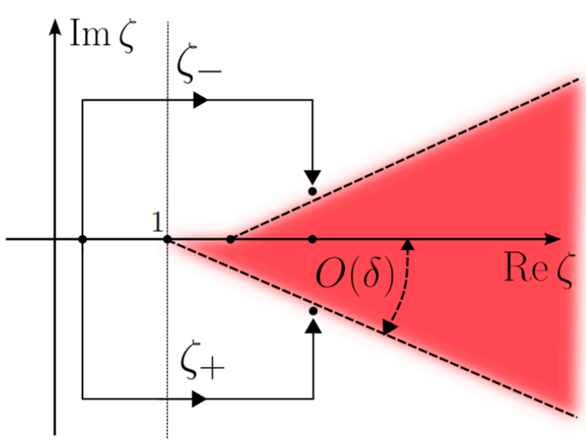
$$K^{\pm}_{\epsilon,\epsilon'|x,t;\beta}(\theta,\theta') = \frac{\pm e^{im\epsilon(x\sinh(\theta+i\epsilon\delta)-t\cosh(\theta+i\epsilon\delta))+\epsilon\eta(\theta+i\epsilon\delta)}}{2\pi\epsilon(1-e^{-\epsilon m\beta\cosh(\theta+i\epsilon\delta)})} \left(\frac{2\epsilon\, e^{\theta+i\epsilon\delta}}{\epsilon\, e^{\theta+i\epsilon\delta} + \epsilon'\, e^{\theta'+i\epsilon'\delta}}\right)$$

It is possible to extract results for time-like separations!

Idea: make $\zeta = t/x$ complex!

Extrapolate results to the physical point

$$\mathrm{Re}\,\zeta_+ = \mathrm{Re}\,\zeta_-, \qquad \mathrm{Im}\,\zeta_+ = \mathrm{Im}\,\zeta_- = 0$$

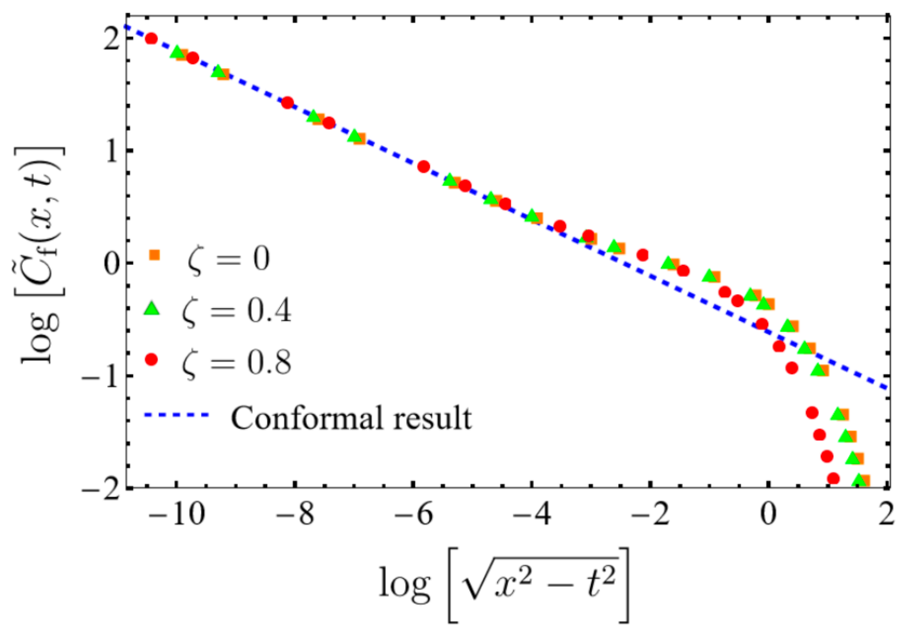


I. Csépányi, M.K., SciPost Phys. **17**, 162 (2024)

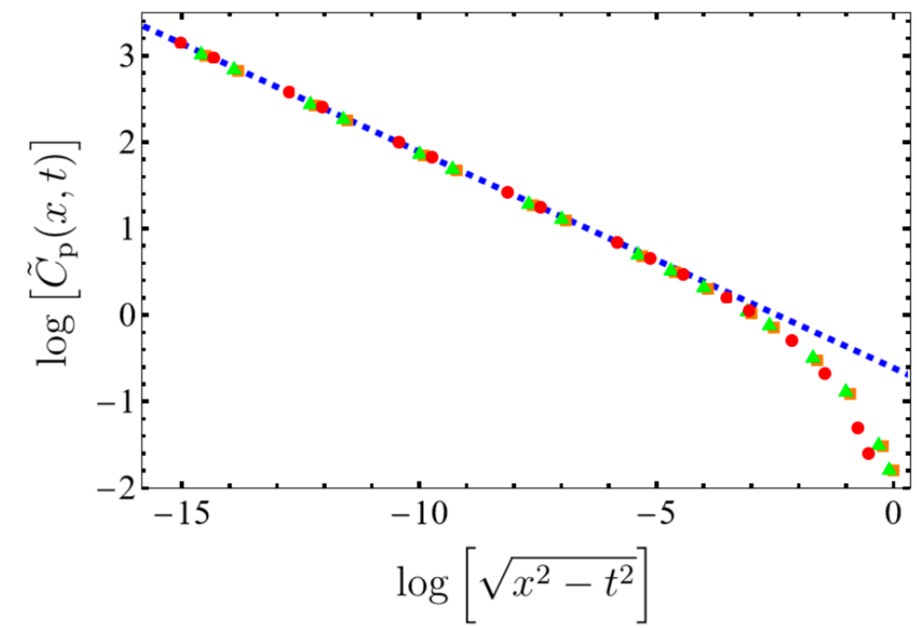
Field theory: small separations

$x, t \ll m^{-1}, \beta$ conformal, Lorentz-invariant $C(x,t) \sim \dfrac{2^{-1/6} e^{1/4} \mathscr{A}^{-3}}{(x^2 - t^2)^{1/8}}$

$\zeta = t/x$



(a) Ferromagnetic phase.

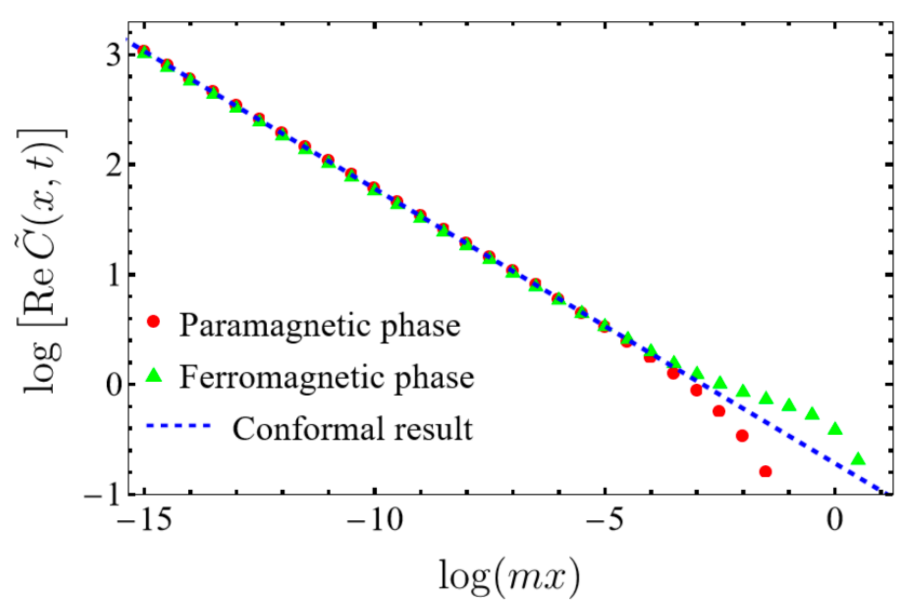


(b) Paramagnetic phase.

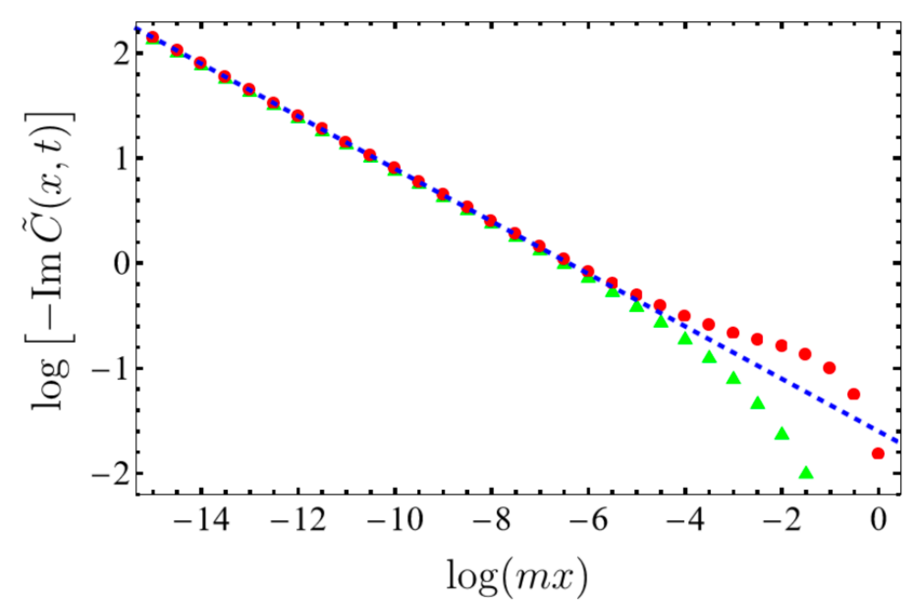
$m\beta = 1$

$\zeta = 1.5$

time-like!



(a) Real part.



(b) Imaginary part.

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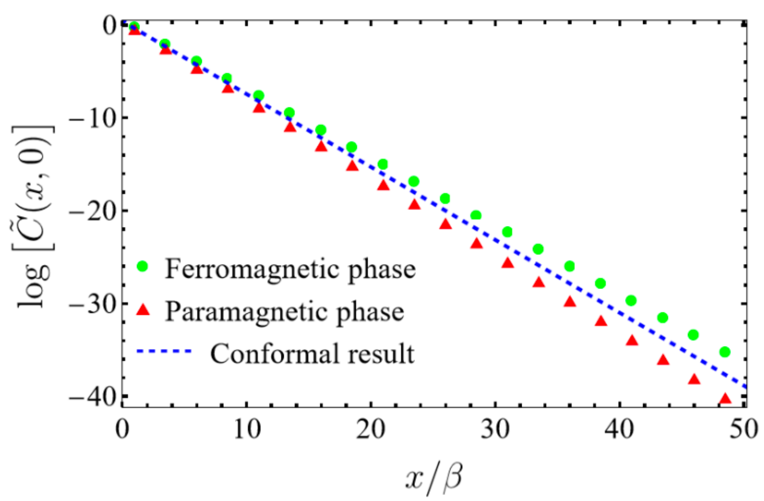
Field theory: high temperature

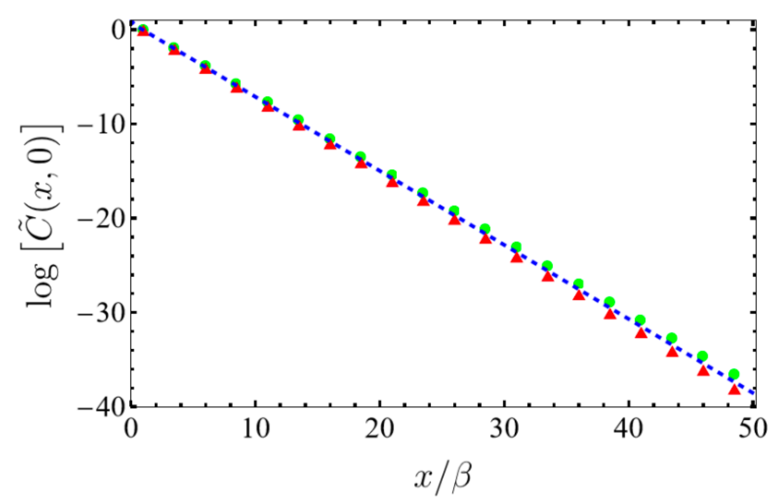
High temperature limit: CFT at finite T

$$C_{\mathrm{f/p}}(x,t;\beta) \xrightarrow{m\beta\to 0} \left(\frac{\pi}{\beta}\right)^{1/4} \frac{2^{-1/6}e^{1/4}\mathscr{A}^{-3}}{\left[\sinh\left(\frac{\pi}{\beta}(x+t)\right)\sinh\left(\frac{\pi}{\beta}(x-t)\right)\right]^{1/8}}$$

equal time



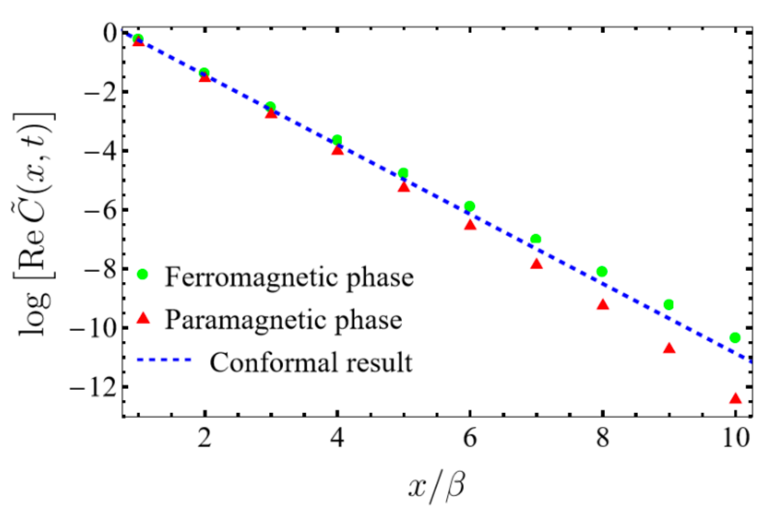
(a) $m\beta = 0.1$.



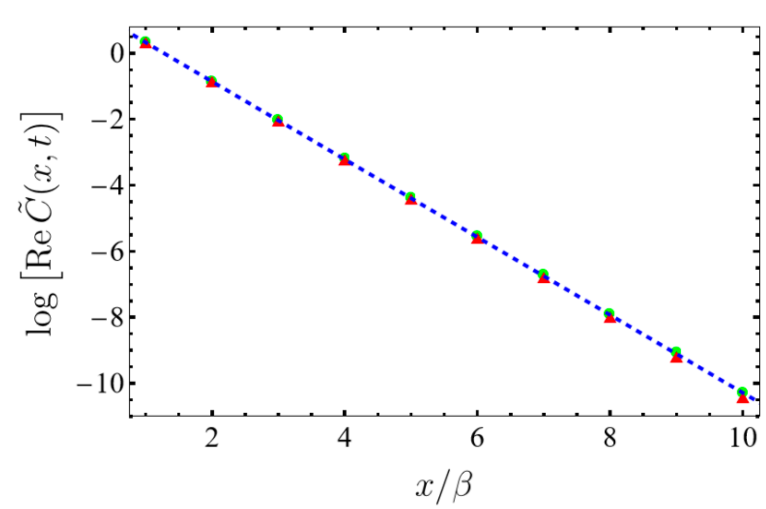
(b) $m\beta = 0.03$.

$\zeta = 1.5$

time-like!



(a) $m\beta = 0.01$.



(b) $m\beta = 0.001$.

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Field theory: asymptotics

Equal time: agreement with the exact result of S. Sachdev, Nucl. Phys. B **464**, 576 (1996).

Time-like separations: agreement with form factor result of Granet, Fagotti, Essler, (2020).

Space-like separations, paramagnetic phase: the first term in the FF series dominates

$$C_{\mathrm{p}}(x,t;\beta) \xrightarrow{x,t\to\infty} C_{\mathrm{p}}^{\mathrm{asym}}(x,t;\beta) = m^{1/4}\,S(m\beta)\,e^{-\Delta\mathscr{E}x}\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\left(\frac{e^{im(x\sinh\theta - t\cosh\theta)}}{1-e^{-m\beta\cosh\theta}}e^{\eta(\theta)} + \frac{e^{-im(x\sinh\theta - t\cosh\theta)}}{e^{m\beta\cosh\theta}-1}e^{-\eta(\theta)}\right)$$

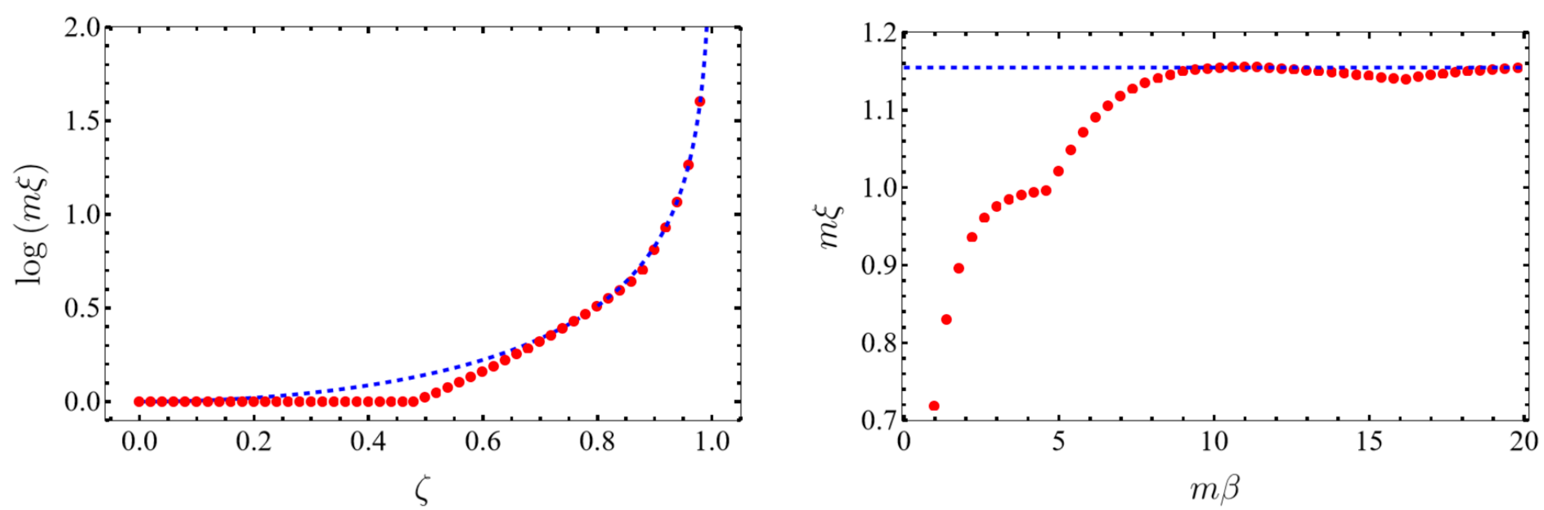
Residue theorem

$$C_{\mathrm{p}}^{\mathrm{asym}}(x,t;\beta) = m^{1/4}\,S(m\beta)\,e^{-\Delta\mathscr{E}x}\sum_{k=-\infty}^{\infty}\frac{\exp\left[-x\sqrt{m^2+q_k^2}+tq_k\right]e^{2\kappa(q_k)}}{\beta\sqrt{m^2+q_k^2}} \qquad q_k = \frac{2\pi k}{m\beta}$$

The slowest decaying term gives the correlation length

$$\xi(\zeta;\beta) = \left[\sqrt{m^2+\frac{4\pi^2}{\beta^2}k_0^2} - \zeta\frac{2\pi}{\beta}k_0 + \Delta\mathscr{E}(\beta)\right]^{-1} \qquad k_0 = \underset{k\in\mathbb{Z}}{\operatorname{argmin}}\left[\sqrt{1+\frac{4\pi^2}{m^2\beta^2}k^2} - \zeta\frac{2\pi}{m\beta}k\right]$$

Non-analytic behavior!



(a) $m\beta = 5$

(b) $\zeta = 0.5$

Non-monotonic temperature dependence!

I. Csépányi, M.K., SciPost Phys. **17**, 162 (2024)

Correlations in the Ising spin chain

with István Csépányi, Giuseppe Del Vecchio Del Vecchio, Benjamin Doyon

Spin chain: the method

$$H = -J\sum_{j=1}^{N}\left(\sigma_j^z\sigma_{j+1}^z + h\sigma_j^x\right) \qquad\qquad C(j,t;\beta) = \langle \sigma_{j+1}^x(t)\,\sigma_1^x(0)\rangle_\beta = ?$$

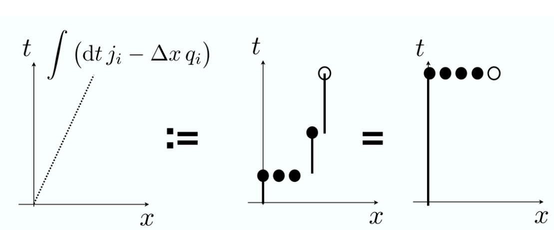
G. Del Vecchio Del Vecchio, B. Doyon: J. Stat. Mech 2022, 053102 (2022): XX spin chain

$$H_{XX} = -J\sum_{j=1}^{N}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y - g\sigma_j^z\right)$$

$$\langle \sigma_j^+(t)\,\sigma_0^-(0)\rangle_\beta \asymp \left\langle e^{i\pi Q(j,t)}\right\rangle \langle c_j^\dagger(t)\,c_0(0)\rangle_{i\pi}$$

string contribution

fermionic correctorrelator in modified state: $\beta\mu = i\pi$



$$Q(x,t) = \int_\gamma q(x',t)dx' - j(x,t')dt'$$

$U(1)$ charge and current density

Ballistic Fluctuation Theory (BFT): J. Myers, M. J. Bhaseen, B. Doyon, SciPost Phys. **8**, 007 (2020)

describes large-scale distributions of integrated charges/currents in integrable models

characteristic function

$$\langle e^{\lambda Q(x,t)}\rangle \asymp e^{x\,F(\lambda;\zeta)} \qquad\qquad \zeta = t/x$$

exact result for $F(\lambda;\zeta)$ Full Counting Statistics in integrable models (language of TBA, GHD)

Spin chain: the method

Ising?

Problem: no conserved U(1) charge!

Solution: doubling trick

$$c_j, c_j^\dagger \qquad\qquad b_j, b_j^\dagger$$

Dirac fermion:

$$\Psi_j = \frac{1}{2}\begin{pmatrix} a_j^{(1)} + i a_j^{(2)} \\ \bar{a}_j^{(1)} + i\bar{a}_j^{(2)} \end{pmatrix}$$

$$a_j^{(1)} = c_j^\dagger + c_j \qquad a_j^{(2)} = b_j^\dagger + b_j$$

$$\bar{a}_j^{(1)} = i(c_j^\dagger - c_j) \qquad \bar{a}_j^{(2)} = i(b_j^\dagger - b_j)$$

Dirac Hamiltonian: $H^{\mathrm{D}} = H^{(1)} + H^{(2)}$

Conserved U(1) charge: $Q = \sum_{j=1}^{L} \Psi_j^\dagger \Psi_j$

importantly, $\prod_{l=0}^{j} e^{i\pi c_l^\dagger c_l} \prod_{l=0}^{j} e^{i\pi b_l^\dagger b_l} = e^{i\pi \sum_{l=1}^{j} \Psi_l^\dagger \Psi_l}$

Squared correlator:

$$\langle \sigma_j^x(t)\, \sigma_0^x(0)\rangle^2 = \langle {}^1\sigma_j^x(t)\; {}^2\sigma_j^x(t)\; {}^1\sigma_0^x(0)\; {}^2\sigma_0^x(0)\rangle^{\mathrm{d}} = \langle a_j^{(1)}(t)\, \bar{a}_0^{(1)}(0)\, a_j^{(2)}(t)\, \bar{a}_0^{(2)}(0)\rangle_{i\pi}^{\mathrm{d}}\, \langle e^{i\pi Q(j,t)}\rangle^{\mathrm{d}} = \langle a_j(t)\, \bar{a}_0\rangle_{i\pi}^2\, \langle e^{i\pi Q(j,t)}\rangle^{\mathrm{d}}$$

$$e^{i\pi \sum_j c_j^\dagger c_j} = e^{i\pi \sum_p \alpha_p^\dagger \alpha_p} = (-1)^F$$

fermionic parity

I. Csépányi et al., arXiv:2602.00794

Spin chain: the result

$$\langle \sigma_x^x(t)\sigma_0^x(0)\rangle \asymp F(x,t,\beta,h)\,\Omega(x,t,\beta,h)$$

Disordered

$$h>1: \quad F(x,t;\beta,h) = -\int_{-\pi}^{\pi}\frac{dp}{2\pi}\left[\frac{e^{i\theta_p}e^{ipx-i\epsilon(p)t}}{1-e^{\beta\epsilon(p)}}+\frac{-e^{i\theta_p}e^{ipx+i\epsilon(p)t}}{1-e^{-\beta\epsilon(p)}}\right]$$

Bose statistics due to $i\pi$!

Ordered

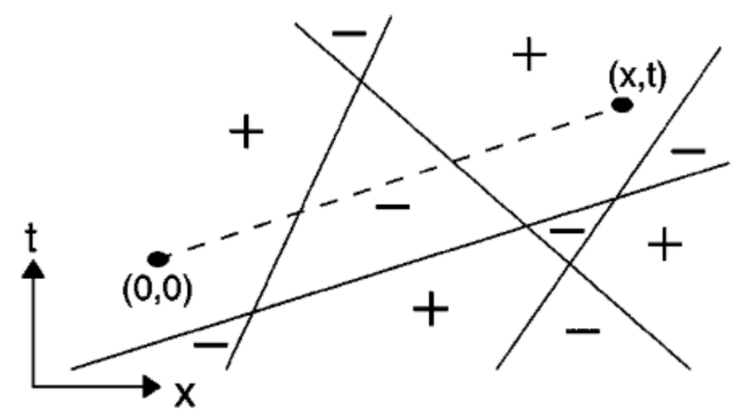
$$h<1: \quad F(x,t;\beta,h) = 1$$

$$\Omega(x,t;\beta,h) = \exp\left[\int_{-\pi}^{\pi}\frac{dp}{2\pi}\,|x-\epsilon'(p)t|\log\left|\tanh\frac{\beta\epsilon(p)}{2}\right|\right]$$

- Low temperature: $\log|\tanh(\beta\epsilon(p)/2)| \approx -e^{-\beta\epsilon(p)}$

  Semiclassical interpretation

  S. Sachdev, A. P. Young, PRL 78, 2220 (1997)



- Form factor calculation: E. Granet, M. Fagotti, F. Essler, SciPost Phys. 9, 033 (2020)

- For $x/t > \max(\epsilon'(p)) = v_{\max}$ $\qquad \Omega(x,t;\beta,h) = \exp\left[x\int_{-\pi}^{\pi}\frac{dp}{2\pi}\log\tanh\frac{\beta\epsilon(p)}{2}\right]$ $\qquad$ independent of t

I. Csépányi et al., arXiv:2602.00794

Spin chain: correlation length for $h > 1$

$$\langle \sigma_x^x(t)\sigma_0^x(0)\rangle \asymp F(x,t,\beta,h)\,\Omega(x,t,\beta,h)$$

$$F(x,t;\beta,h) = -\int_{-\pi}^{\pi}\frac{dp}{2\pi}\left[\frac{e^{i\theta_p}e^{ipx-i\epsilon(p)t}}{1-e^{\beta\epsilon(p)}} + \frac{-e^{i\theta_p}e^{ipx+i\epsilon(p)t}}{1-e^{-\beta\epsilon(p)}}\right] = \int_C \frac{dz}{2\pi}\frac{e^{i\theta_z}e^{izx-i\epsilon(z)t}}{1-e^{\beta\epsilon(z)}}$$

C : complex contour on a double-sheeted Riemann surface (branches of square root)

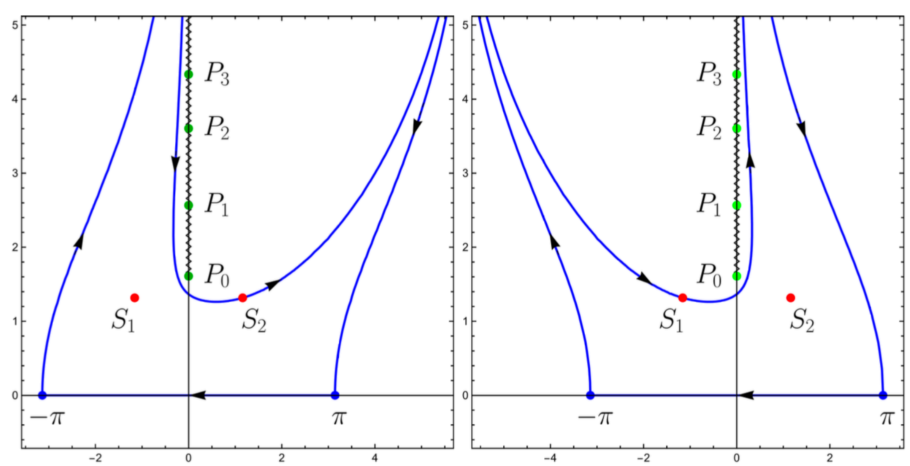
$\zeta = t/x > 1$ Time-like regime. Stationary phase: critical points are real $\longrightarrow$ algebraic decay

$1/h < \zeta < 1$ Saddle point method

$$F(x,t) \asymp \frac{e^{-x/\xi_{\rm f}}}{\sqrt{x}} \times \text{oscillatory terms}$$

$$\xi_{\rm f}^{-1} = -\sqrt{1-\zeta^2} + \operatorname{arccosh}(1/\zeta)$$

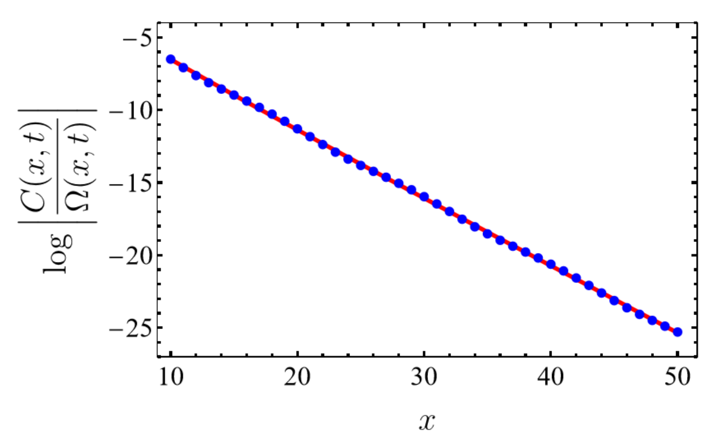
independent of h, β

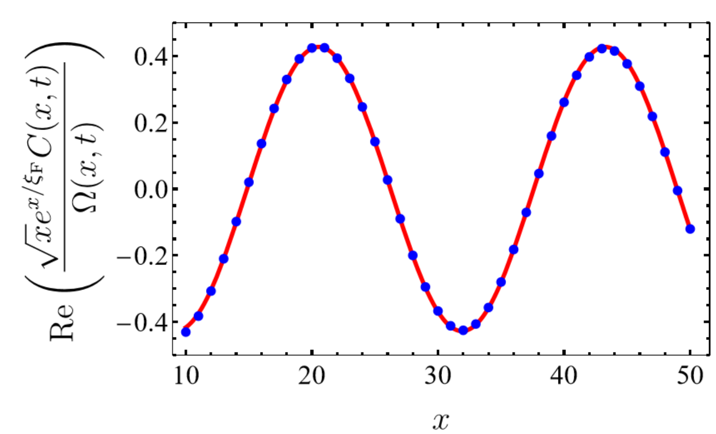


(a) Contour for $I^{(1)}$. (b) Contour for $I^{(2)}$.

Comparison with first principles numerics (Pfaffian)

$h = 3, \zeta = 0.5, \beta = 1$





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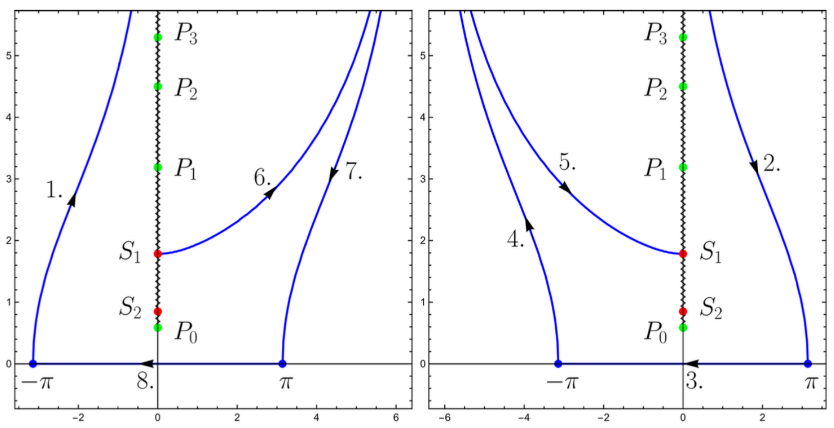
Spin chain: correlation length

Paramagnetic phase, space-like regime

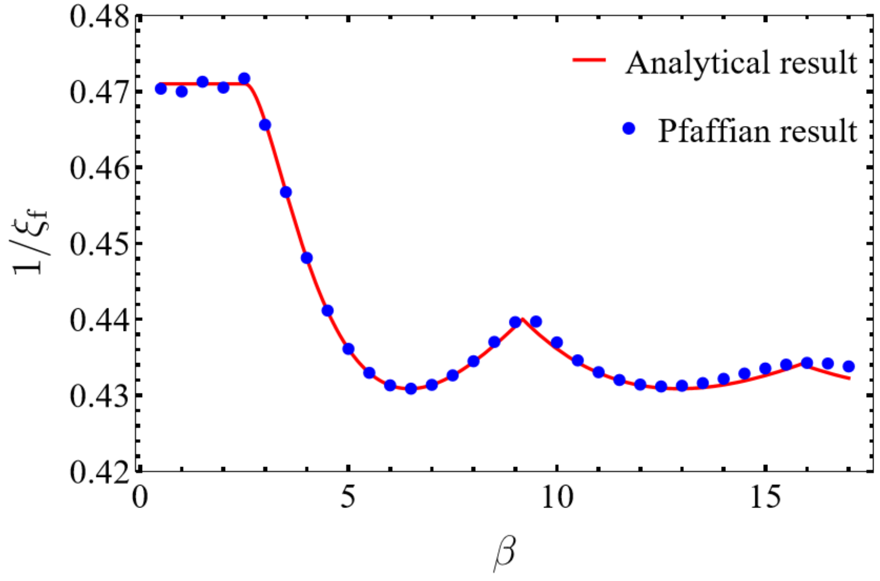
$\zeta < 1/h < 1$

$$\xi_{\mathrm{f}}^{-1} = \max\left(\{\Phi^{(1)}(z_{P_j})\}, \Phi^{(1)}(z_{S_1})\right)$$

$$\Phi^{(1)}(z) = iz - i\epsilon(z)\zeta$$

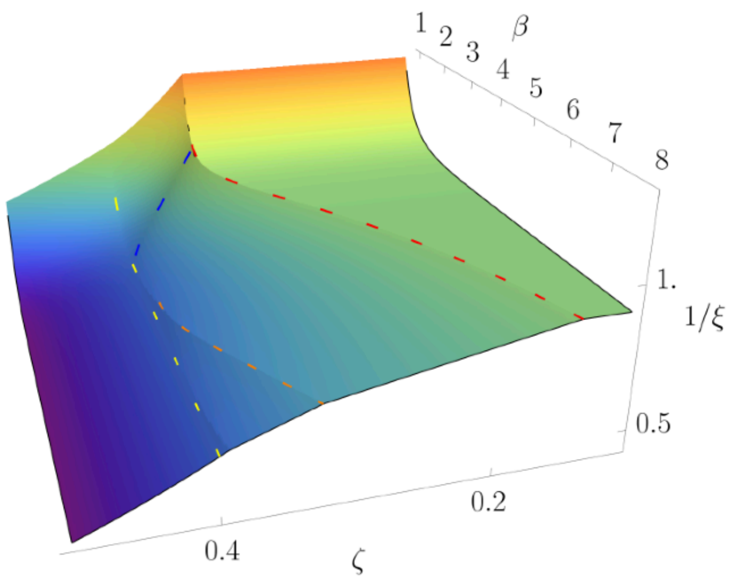


$h = 1.85, \zeta = 0.5$



Non-analytic, non-monotonic!

$h = 2.5$



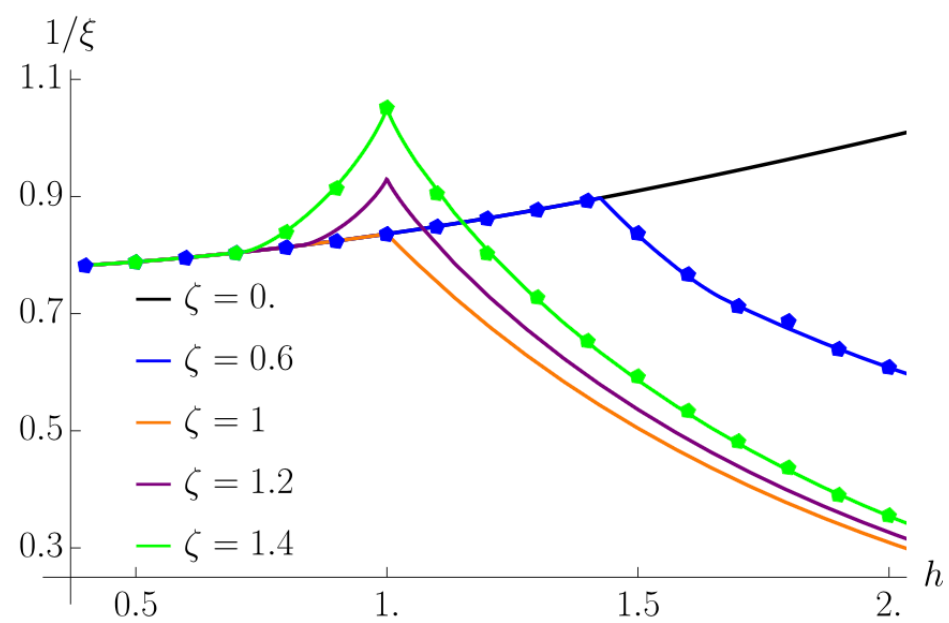
Missed by previous studies, the "fermionic part" was too simple:

- Semiclassics: zero-temperature free-fermion propagator
- form factors: a plausible-looking low-temperature expansion (FD statistics)

But the limits $T \to 0$ and $x, t \to \infty$ do not commute!

I. Csépányi et al., arXiv:2602.00794

Spin chain: correlation length



$\beta = 1$

$$\frac{d}{dh}\left(\frac{1}{\xi^{\textbf{time}}}\right) \sim \frac{\sqrt{\zeta^2 - 1}}{\pi\zeta}\mathrm{sgn}(1-h)\log|h-1| \quad (h \to 1)$$

Sensitive to the QCP at any temperature!
Time-like correlation times can be used to detect QCP.

$$\langle \sigma_j^x(t)\sigma_k^x(0)\rangle \asymp F(x,t,\beta,h)\,\Omega(x,t,\beta,h)$$

Propagation - q. particle creation/annihilation
(Sachdev: quantum)

Fluctuations
(Sachdev: classical)

I. Csépányi et al., arXiv:2602.00794

Hydrodynamic language $\longrightarrow$ General lesson?

Summary

- dynamical correlation functions of the order parameter at finite temperature
- scaling limit: Ising field theory
  - numerically exact results in both phases for any temperature and direction
    time-like regime is not fully accessible
  - agreement with various theoretical predictions
  - non-analyticities in the correlation length
- spin chain
  - hydrodynamic approach for large separations
  - non-analyticities in the correlation length
  - correlation length (time) is sensitive to the QCP
- Outlook:
  - physical interpretation of the non-analytic points and cusps
  - universality? other models...

Thank you for your attention!

I. Csépányi et al., arXiv:2602.00794

I. Csépányi, M.K., SciPost Phys. **17**, 162 (2024)