Extracting Filtered Signal Statistics of Continuously Measured Quantum Systems

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30/04/26 — QOPEN Kick-off Workshop, Lisbon

A. Kiely and G. T. Landi *Phys. Rev. A* **113**, 032442 (2026)

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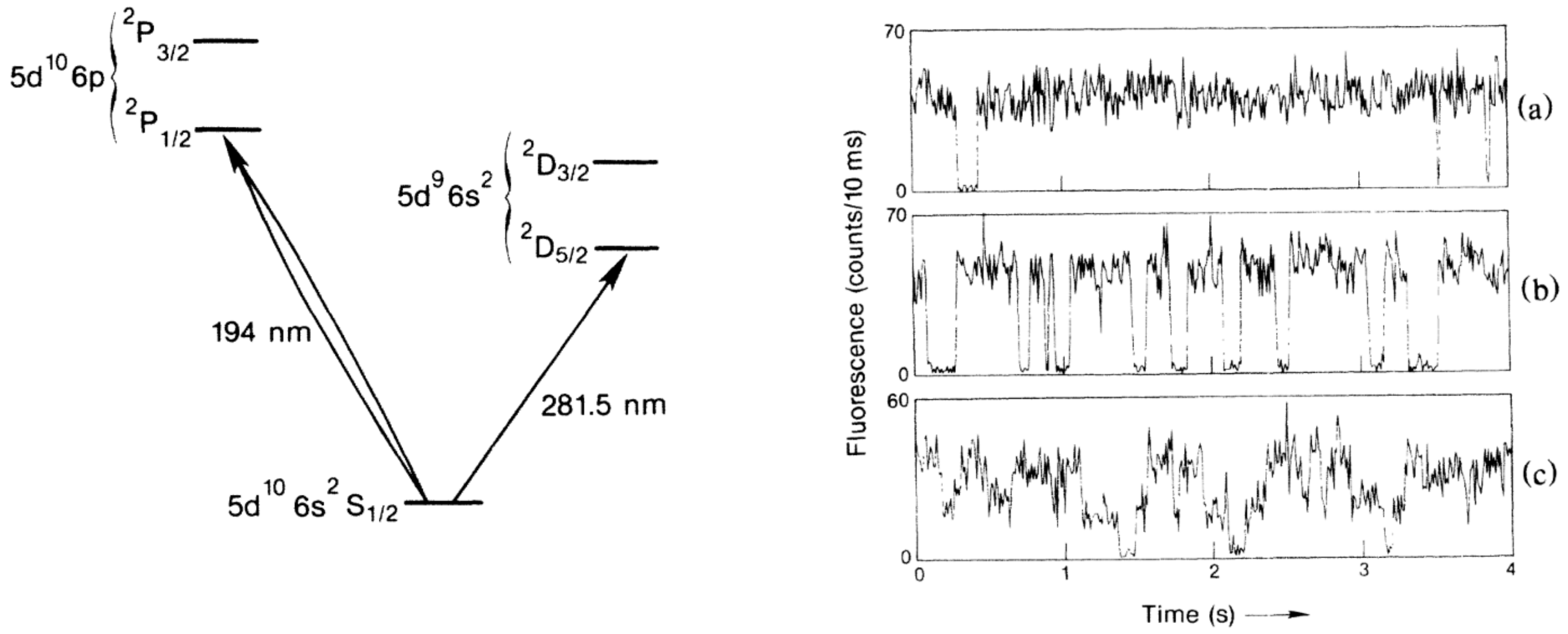
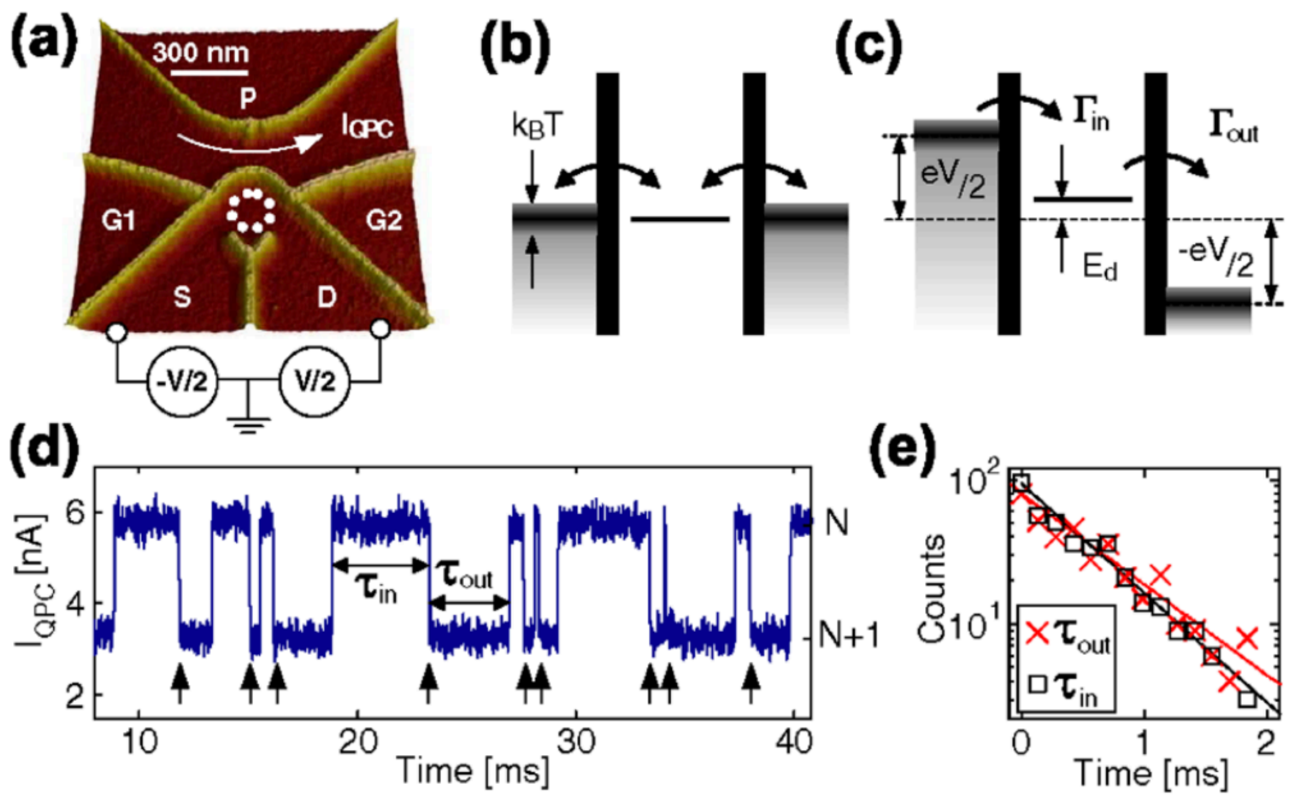
Continuous Quantum Measurements

Atomic transitions



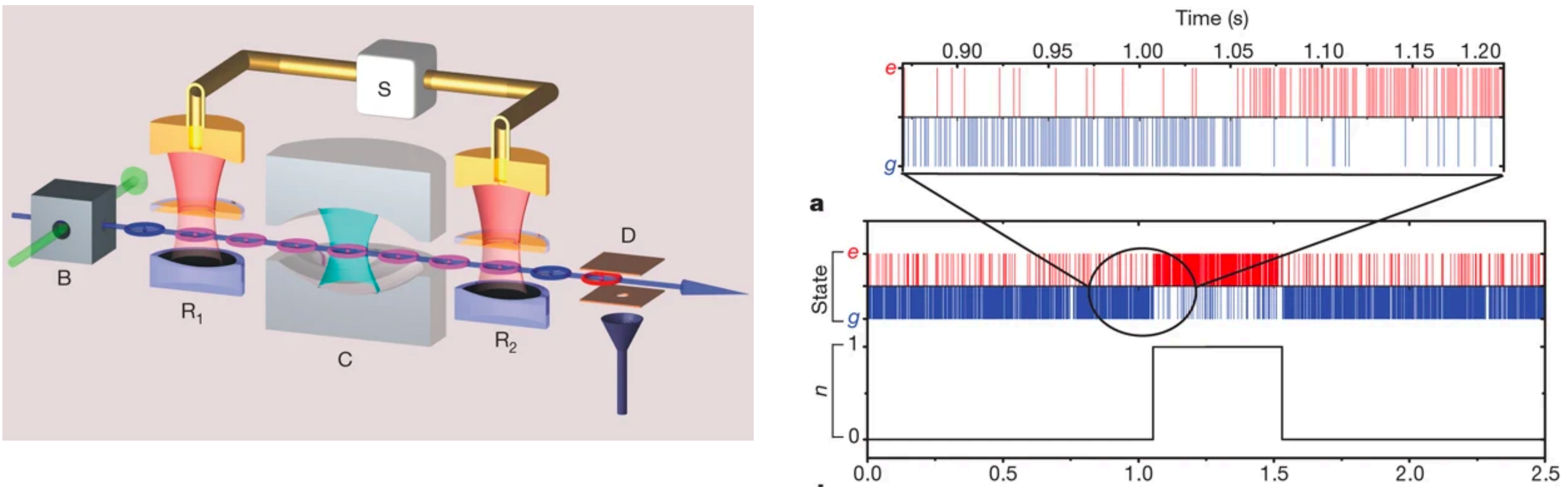
J. C. Bergquist et al., Phys. Rev. Lett. 57, 1699 (1986)

Quantum dots



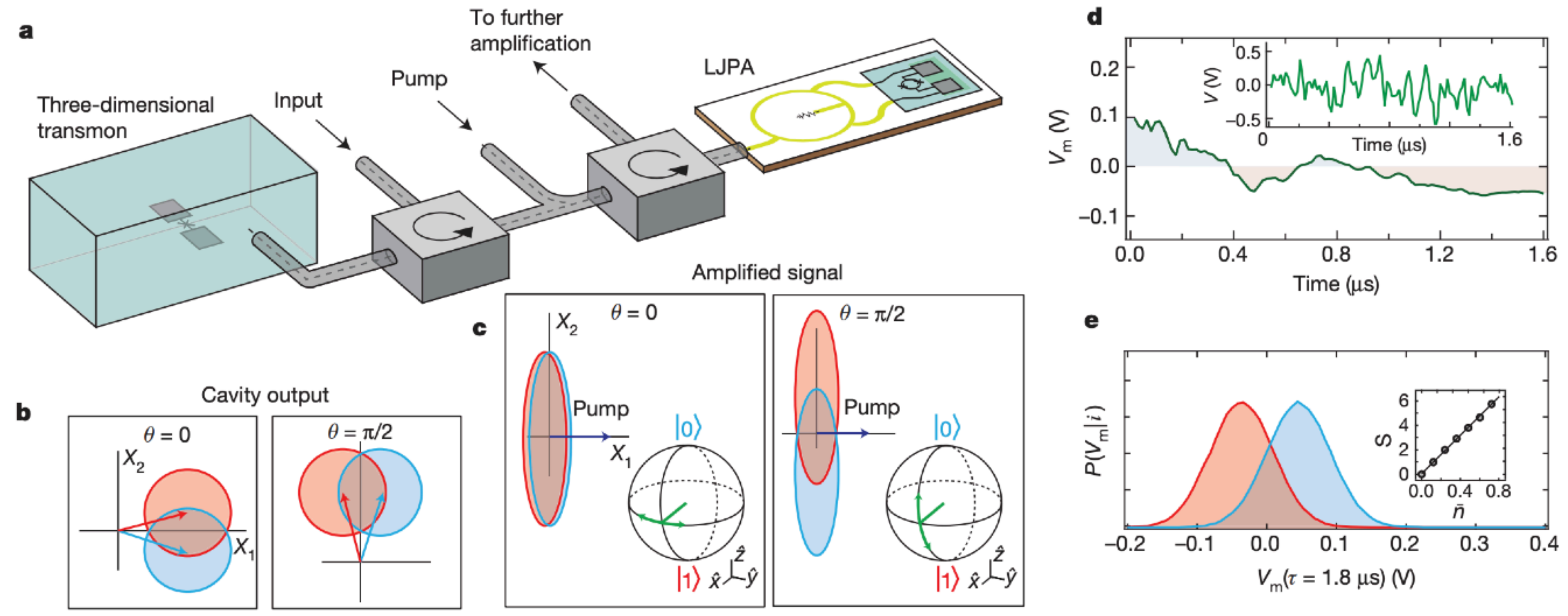
S. Gustavsson et al., Phys. Rev. Lett. 96, 076605 (2006)

Photons in a cavity



S. Gleyzes et al., Nature 446, 297 (2007)

Superconducting circuits

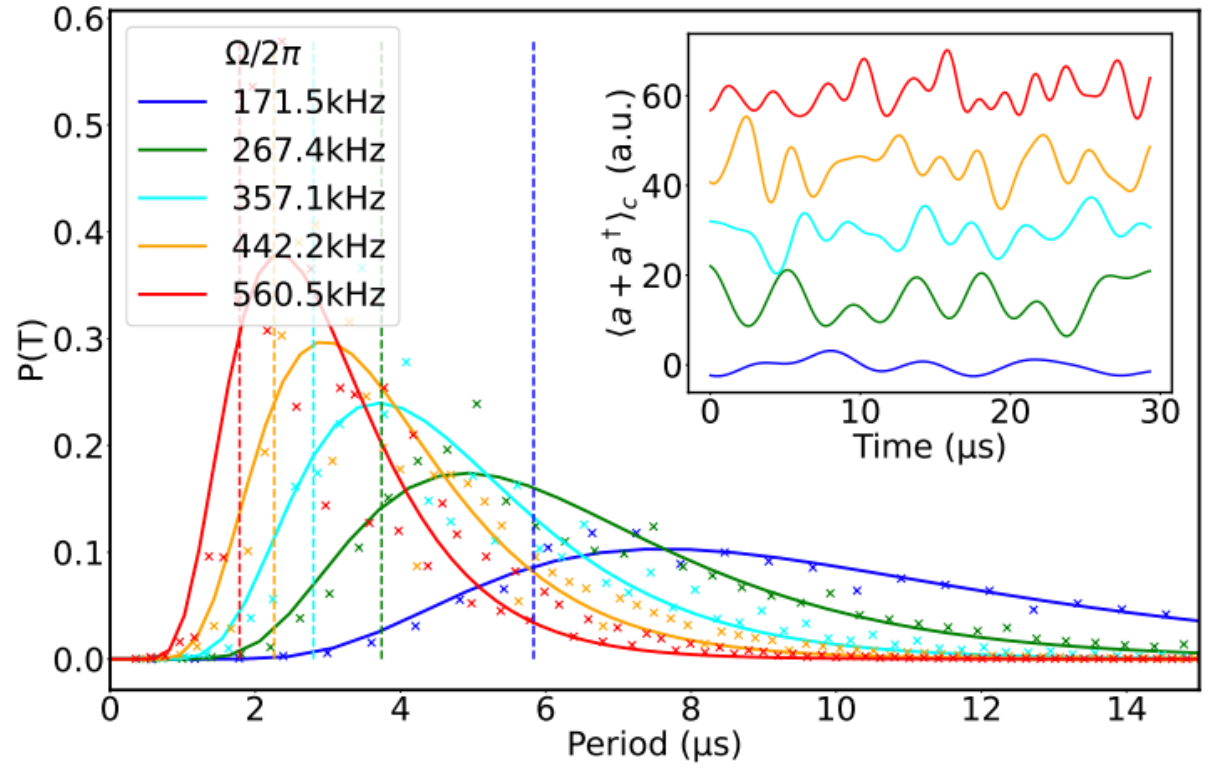


K. Murch et al., Nature 502, 211 (2013)

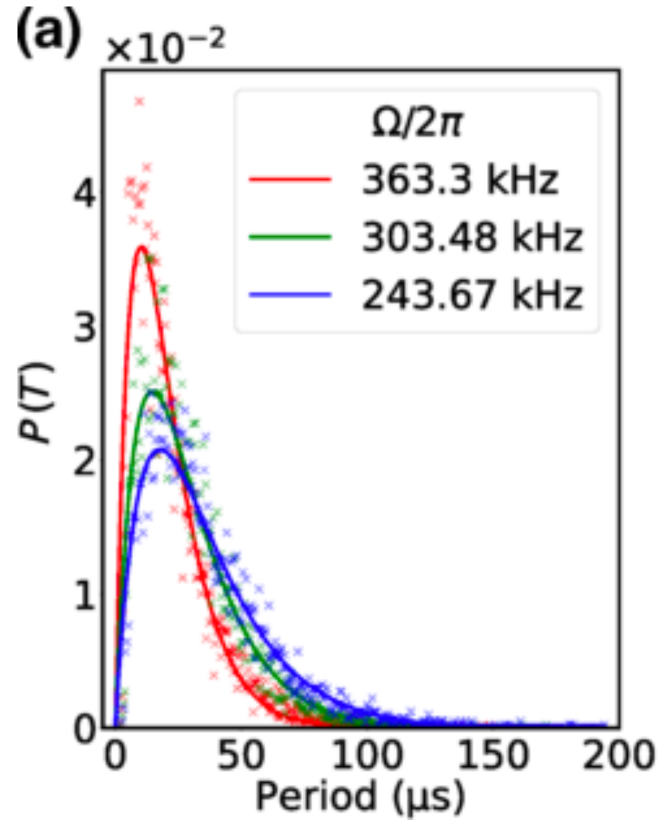
Motivation

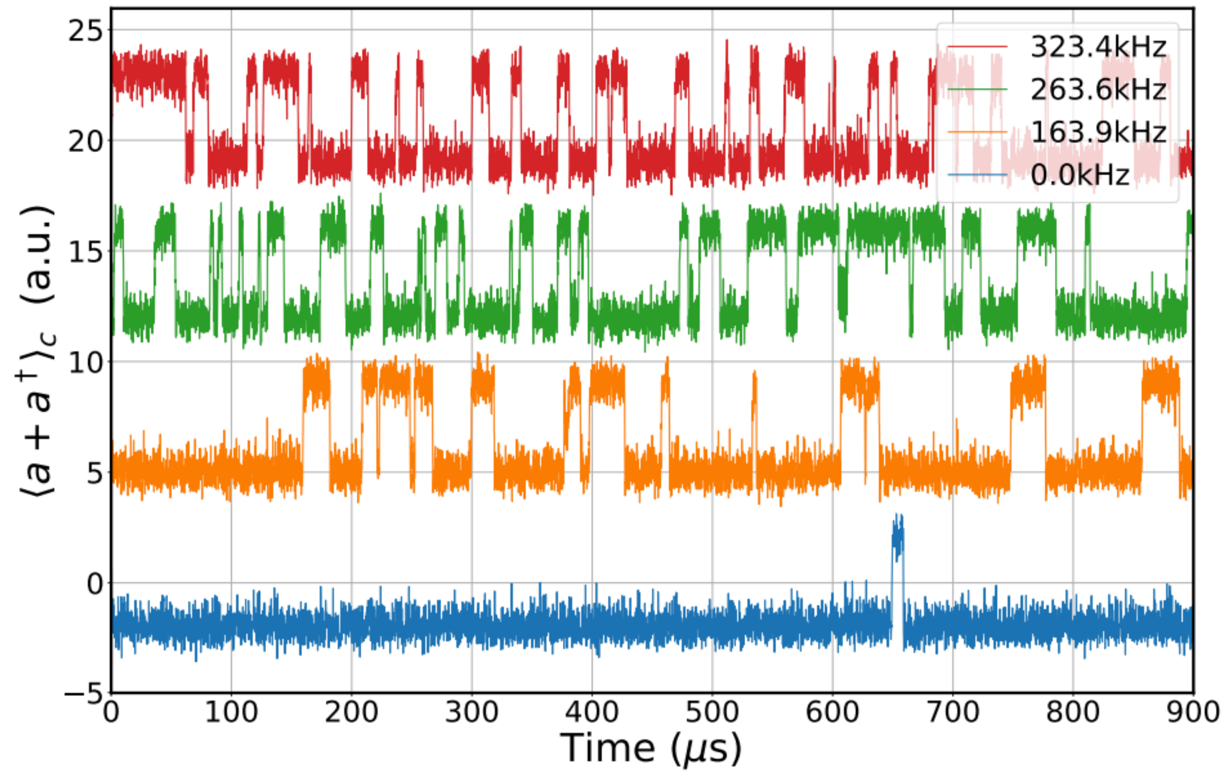
Quantum Clock Experiment

X. He et al., Phys. Rev. Applied 20, 034038 (2023)



Weak measurement regime



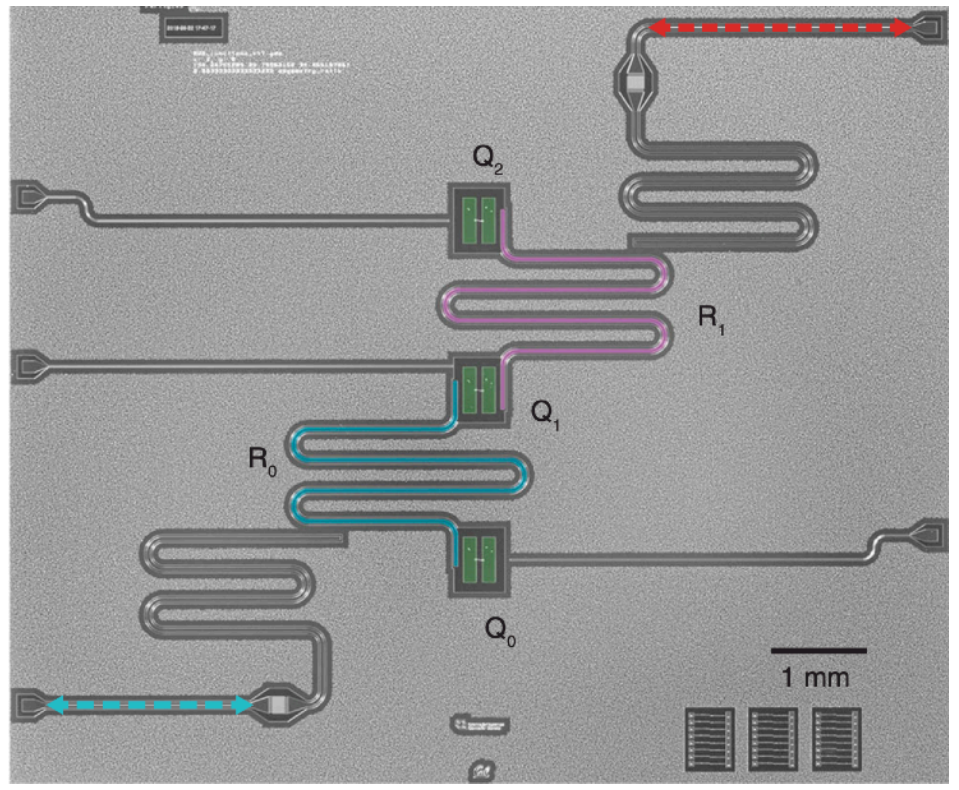


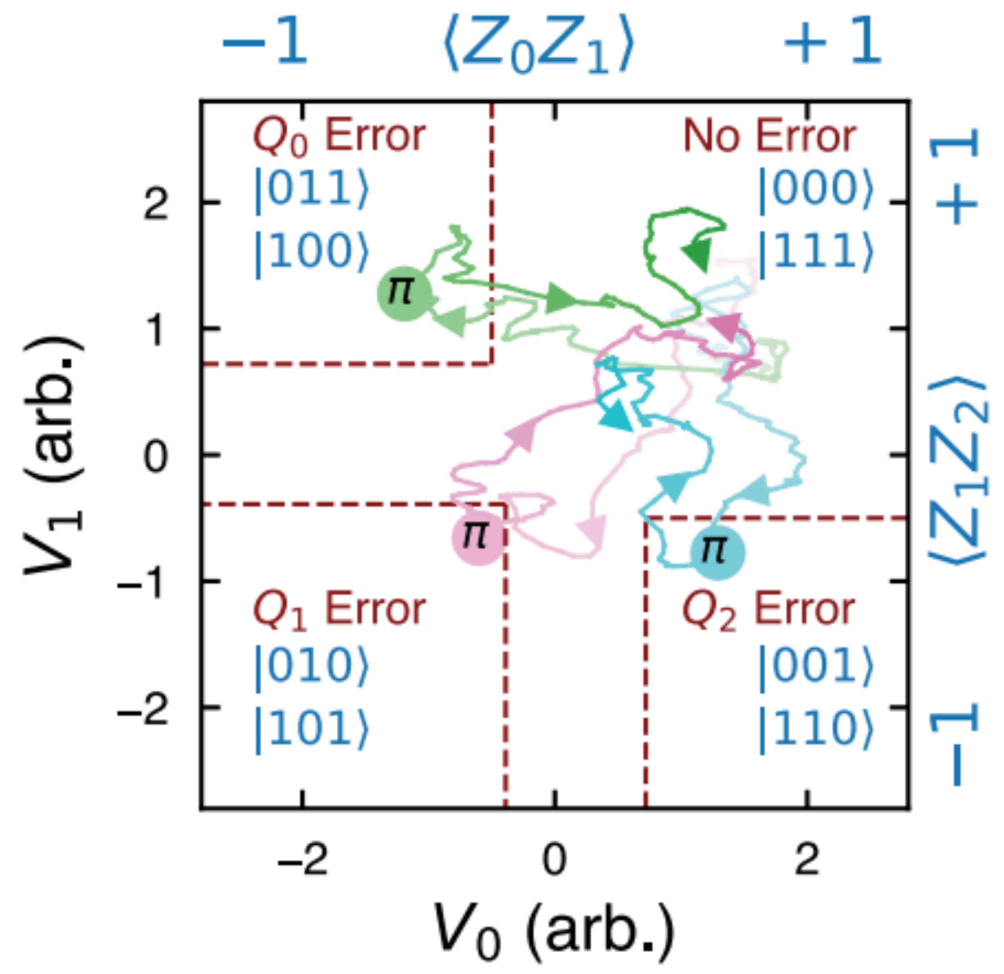
Strong measurement regime

Continuous Quantum Error Correction

W.P. Livingston et al., Nat. Commun. 13, 2307 (2022)

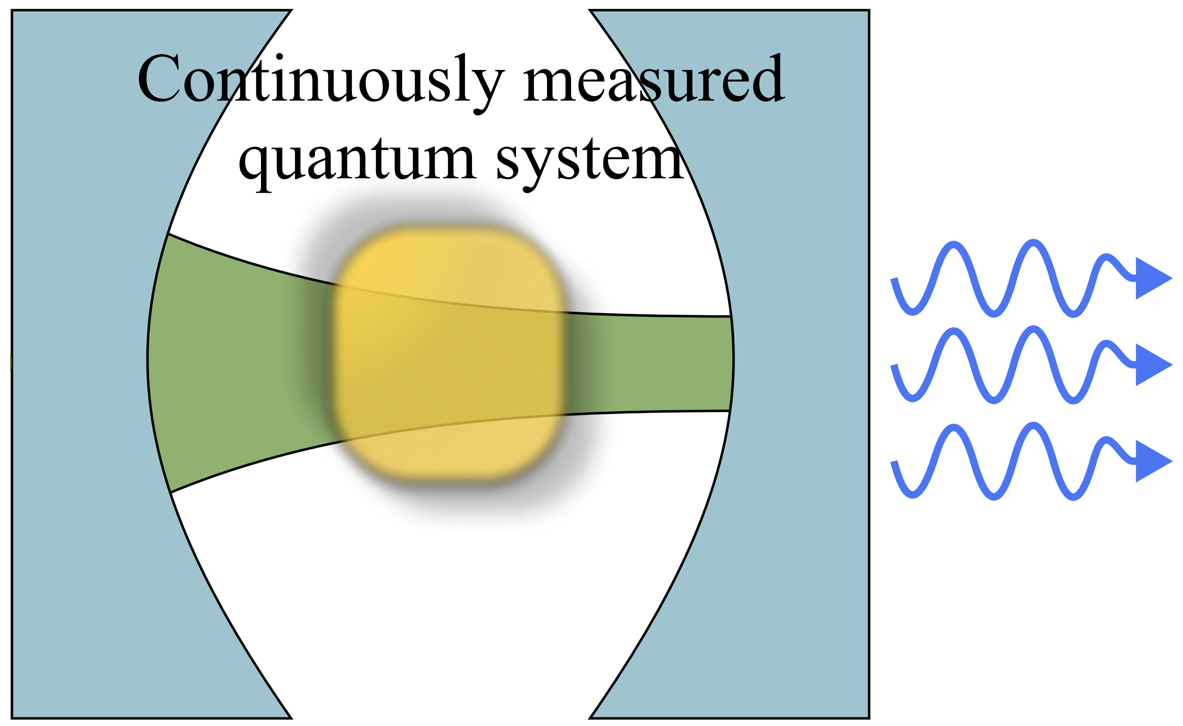
Three qubits in two cavities

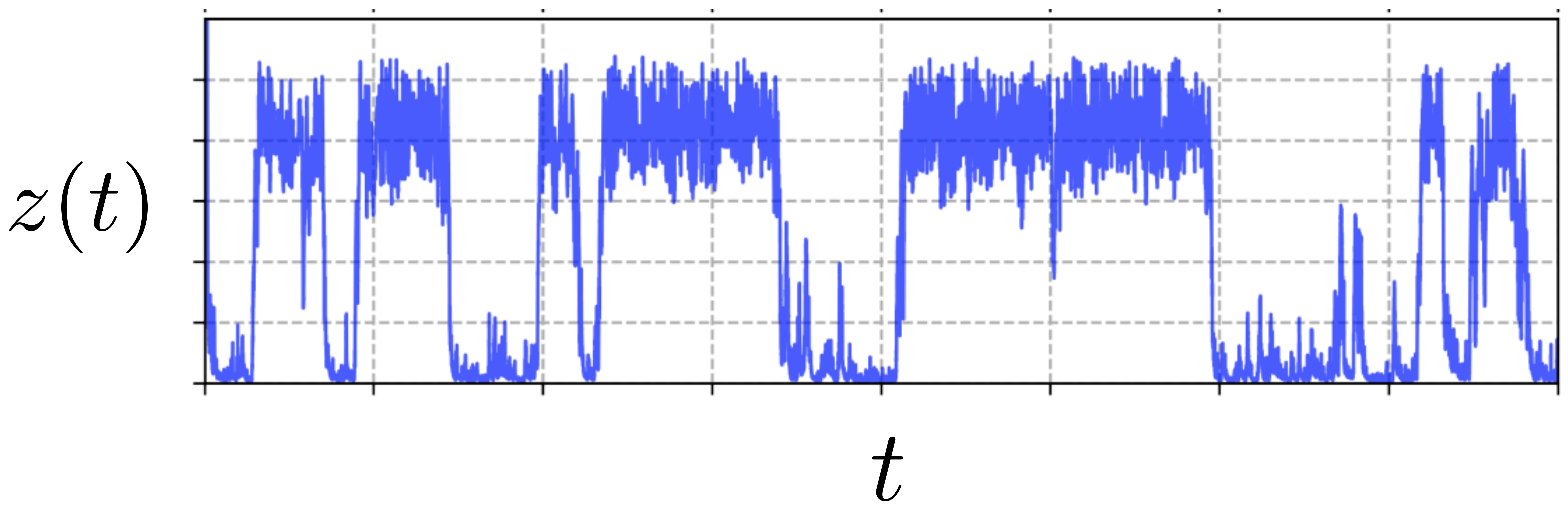




Two continuous measurement signals

Standard Mathematical Description





Stochastic master equation

$$d\rho_c = \mathcal{L}\rho_c dt + \lambda \mathcal{D}[A]\rho_c dt + \sqrt{\lambda}\{A - \langle A \rangle_c, \rho_c\} dW$$

Measurement record

$$z(t) = \langle A \rangle_c + \frac{1}{\sqrt{4\lambda}} \frac{dW}{dt}$$

Conditional density matrix

Stochastic part

Charge Resolved Approach

PHYSICAL REVIEW LETTERS **129**, 050401 (2022)

Quantum Fokker-Planck Master Equation for Continuous Feedback Control

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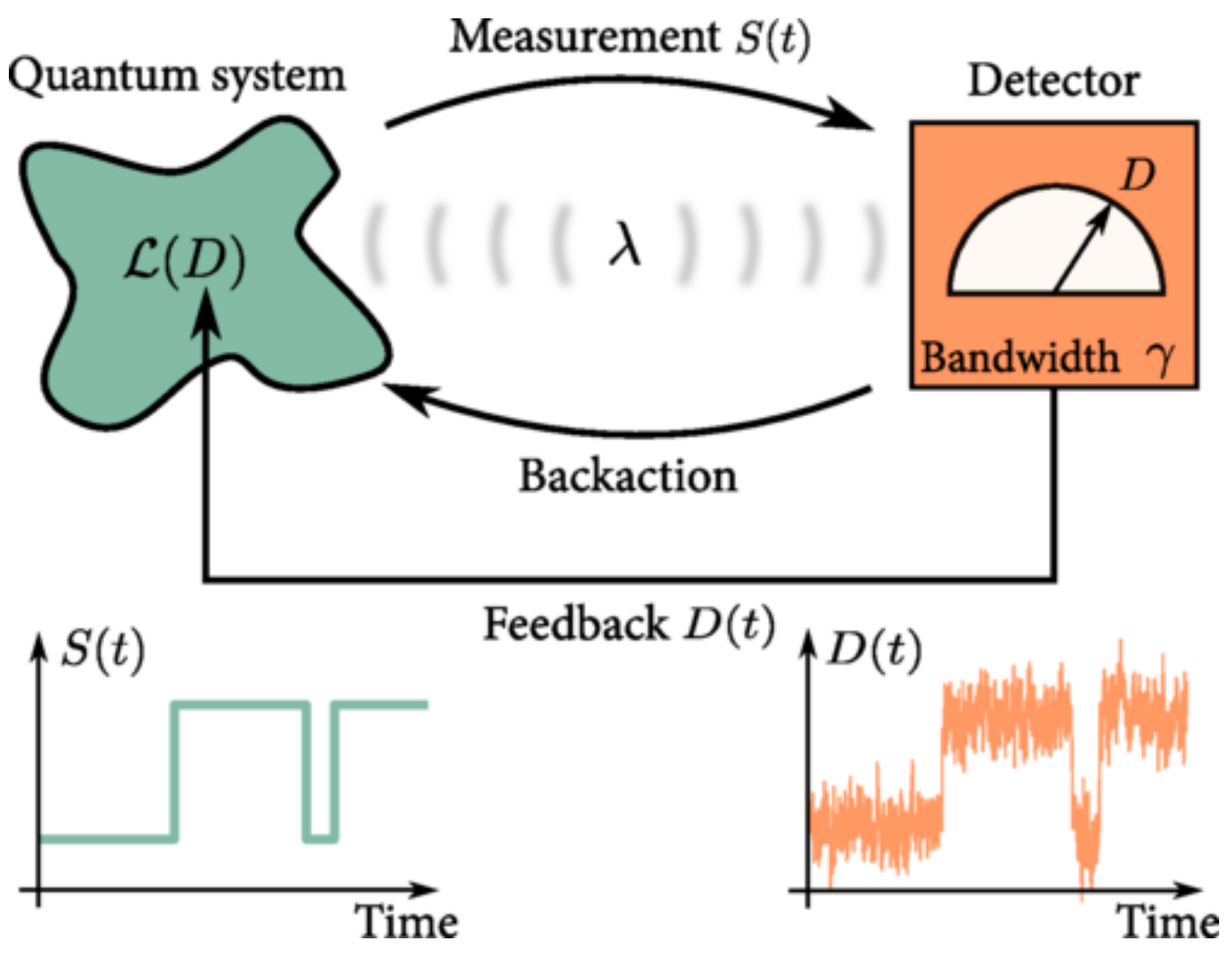


Continuous Measurements

$$d\rho_c = \mathcal{L}\rho_c dt + \lambda\mathcal{D}[A]\rho_c dt + \sqrt{\lambda}\{A - \langle A\rangle_c, \rho_c\}dW$$

Filtered output current

$$D(t) = \int_0^t dt'\, \gamma e^{-\gamma(t-t')} z(t')$$

Joint state

$$\rho_t(D)$$

Detector probability

$$\mathrm{Tr}\{\rho_t(D)\} = P(D,t)$$

Reduced state

$$\int dD\, \rho_t(D) = \rho_t$$

Fokker-Planck Equation

$$\partial_t \rho_t(D) = \underbrace{\mathcal{L}(D)\rho_t(D)}_{\text{Feedback Liouvillian}} + \underbrace{\lambda \mathcal{D}\left[A\right] \rho_t(D)}_{\text{Measurement Backaction}} - \underbrace{\gamma \partial_D \mathcal{A}(D) \rho_t(D)}_{\text{Drift}} + \underbrace{\frac{\gamma^2}{8\lambda} \partial_D^2 \rho_t(D)}_{\text{Diffusion}}$$

Limiting cases

$$D(t) = \int_0^t dt' \, \gamma e^{-\gamma(t-t')} z(t')$$

$\gamma \to \infty$ Instantaneous Current

Phys. Rev. Lett. 70, 548 (1993)

$\gamma \to 0$ Integrated Current

Phys. Rev. A 109, L050202 (2024)

"However, solving Eq. (398) [the Fokker-Planck equation] is generally challenging

— PRX Quantum 5, 020201 (2024)

Linear Equation

$$\partial_t \vec{M} = \hat{Q} \vec{M}$$

$$\rho_t(D) = \sum_{n=0}^{\infty} M_n \, f_n^\sigma(D)$$

$$\vec{M} = (M_0 \, M_1 \, M_2 \ldots M_{N-1})^T$$

Efficient Numerical Solutions

Assumptions

Steady state $\partial_t \vec{M} = \hat{Q}\vec{M} = 0$

No feedback $\mathcal{L}(D) = \mathcal{L}_0$

Block Structure

$$\hat{Q} = \begin{bmatrix} X_1 & 0 & 0 & \cdots & 0 \\ Y_1 & X_2 & 0 & \cdots & 0 \\ 0 & Y_2 & X_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & Y_{n-1} & X_n \end{bmatrix}$$

Forward Substitution

$$(\mathcal{L}_0 + \lambda\mathcal{D}[A])M_0 = 0$$

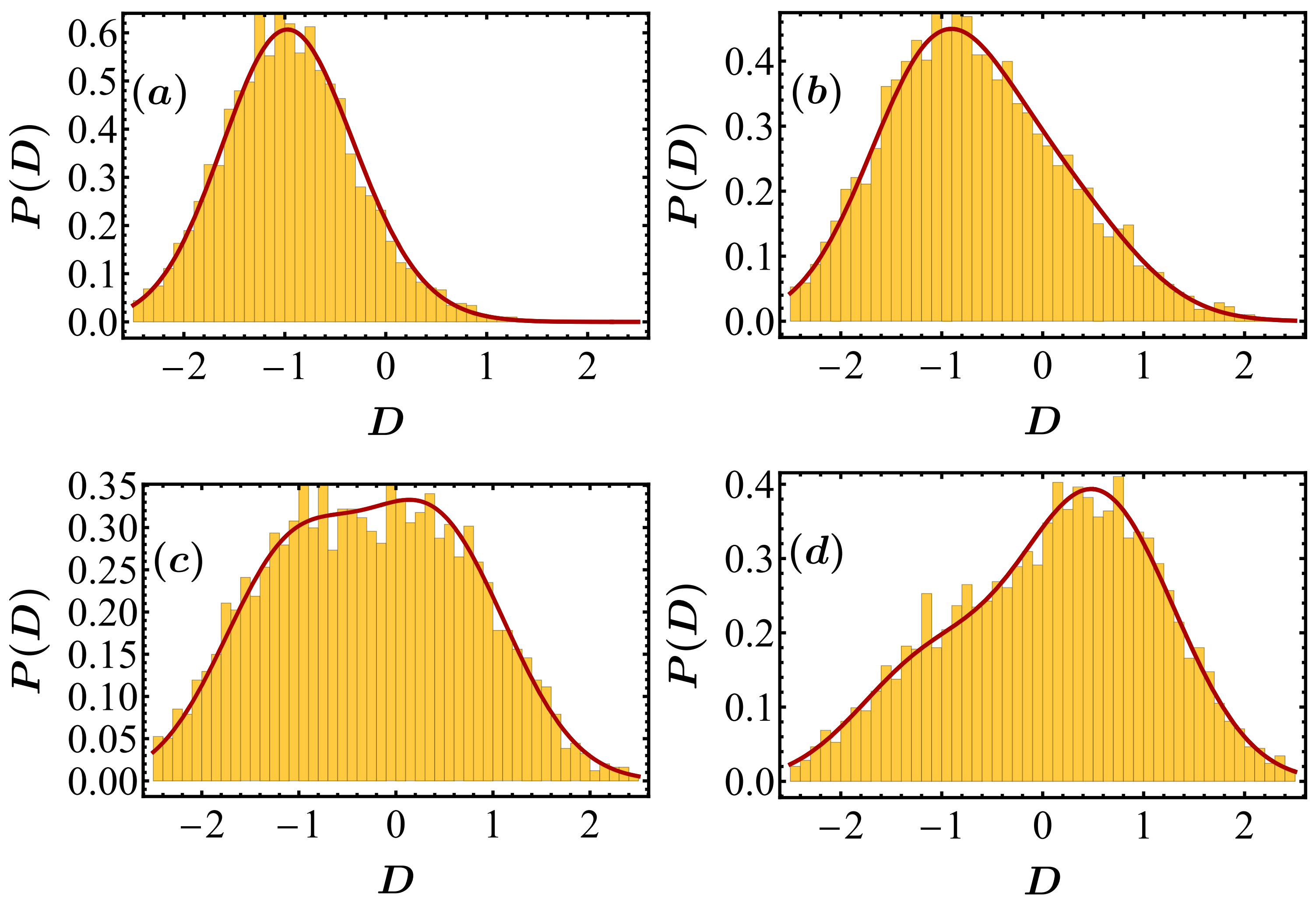
$$X_{n+1}M_n = -Y_n M_{n-1}$$

Reconstruct Characteristic Function

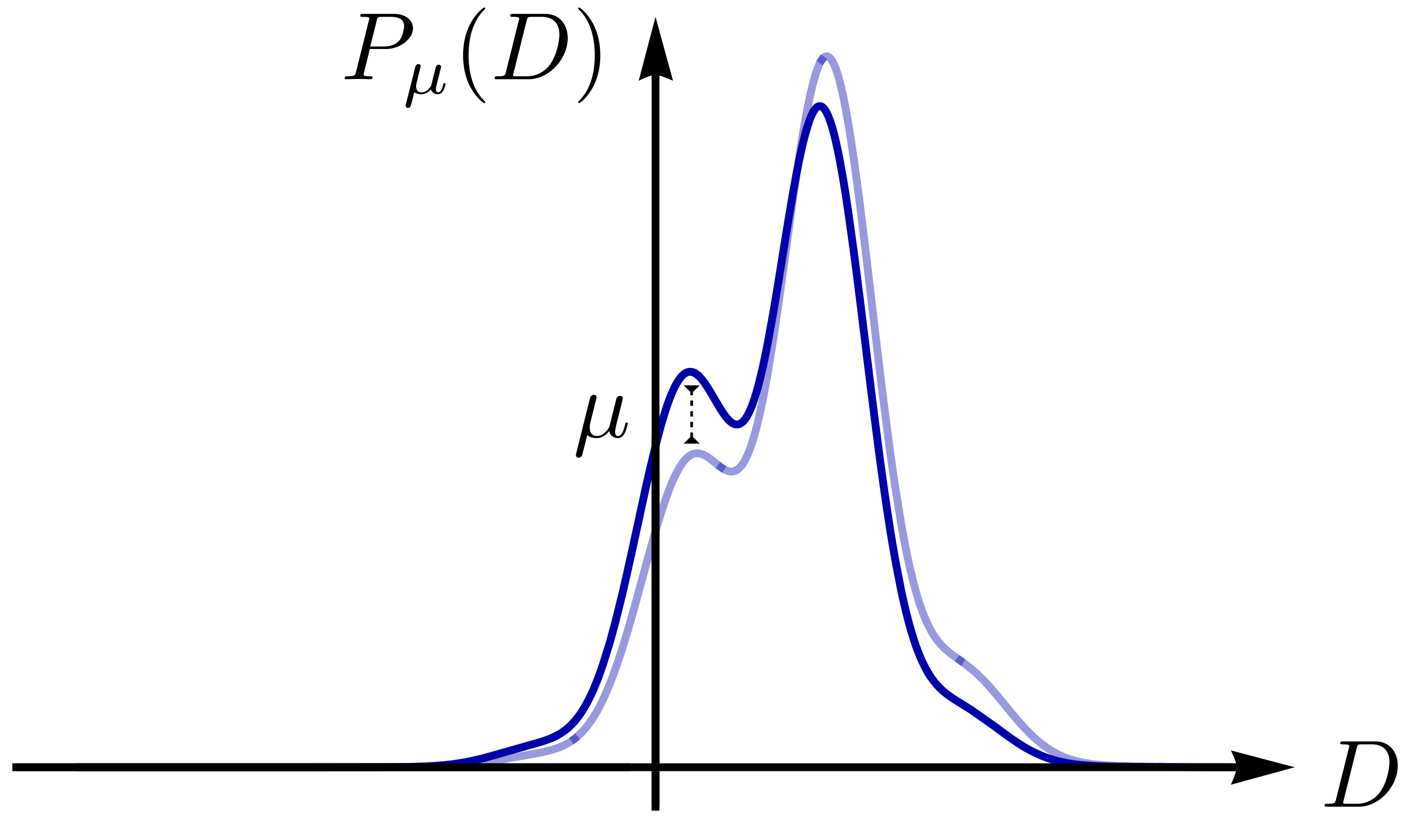
$$\int dD\, e^{iKD}\rho(D) = e^{-K^2\sigma/2}\sum_{m=0}^{M} M_m \frac{(iK)^m\sigma^{m/2}}{\sqrt{m!}}$$

Charge Resolved Approach Applications

Measurement Statistics

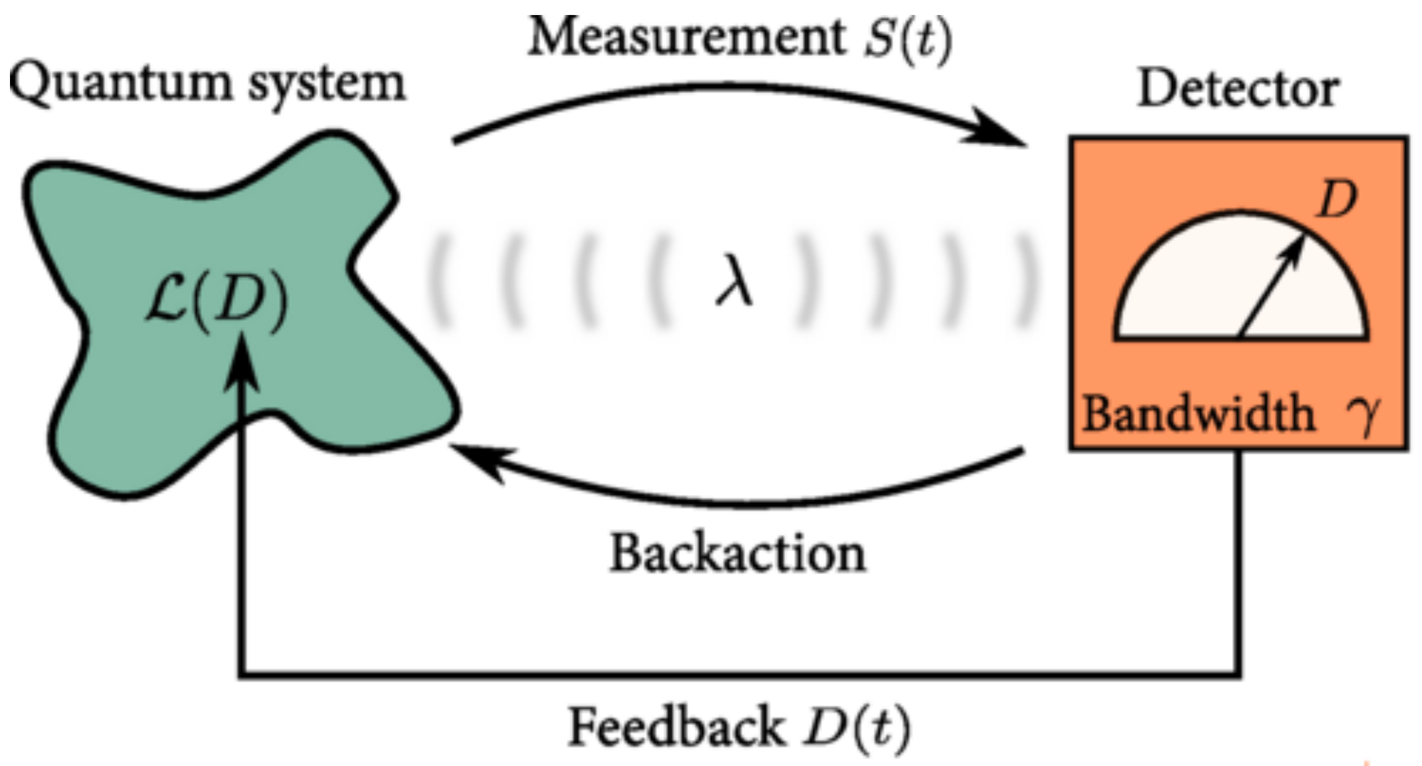


Metrology

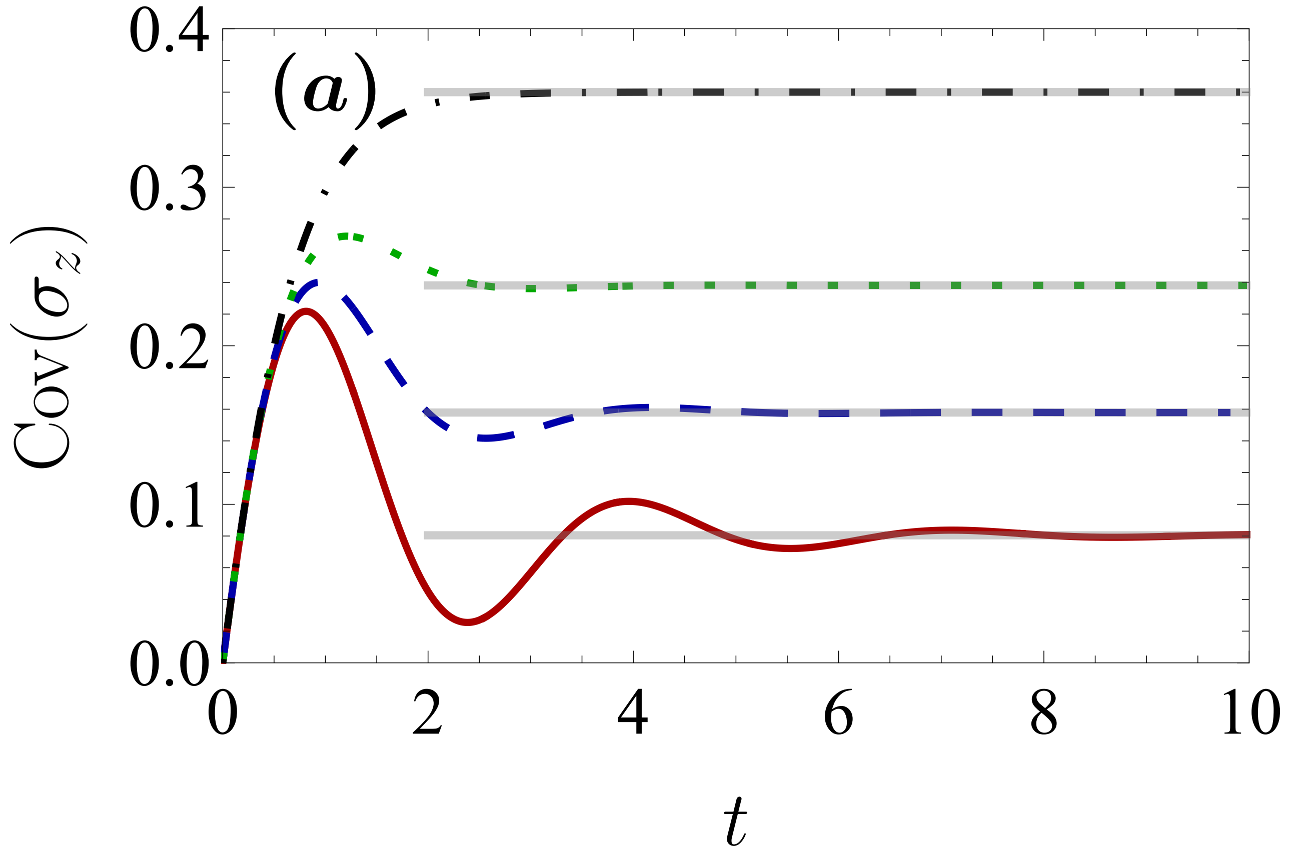


$$F_I = \int dD P_\mu(D) \left|\partial_\mu \ln P_\mu(D)\right|^2$$

Feedback Dynamics



Growth of Correlations



Qubit Example

Hamiltonian $$H = \Omega \sigma_x$$

Observable $$A = \sigma_z$$

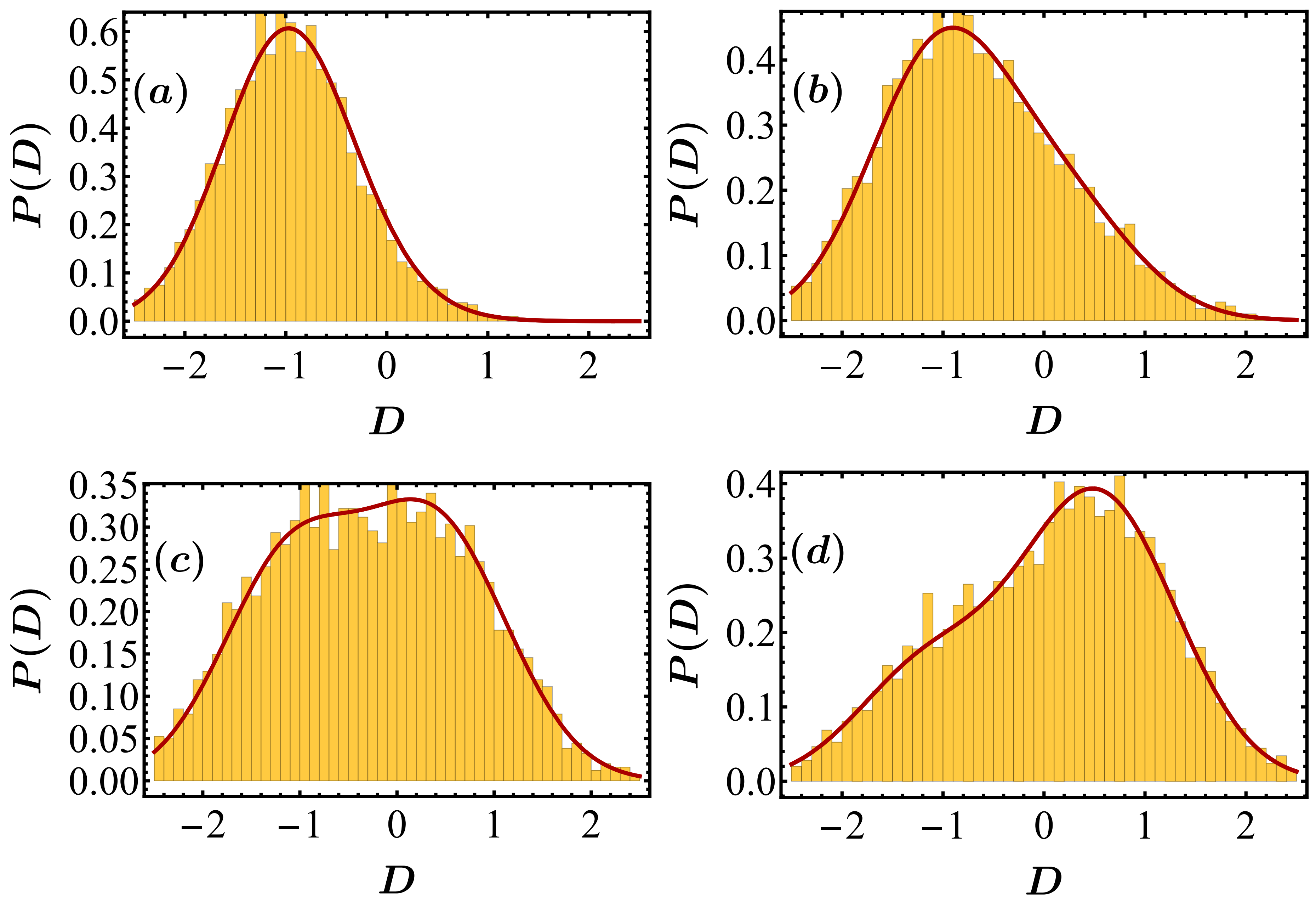
Distribution Construction

$$c_m = \mathrm{Tr}(M_m)$$

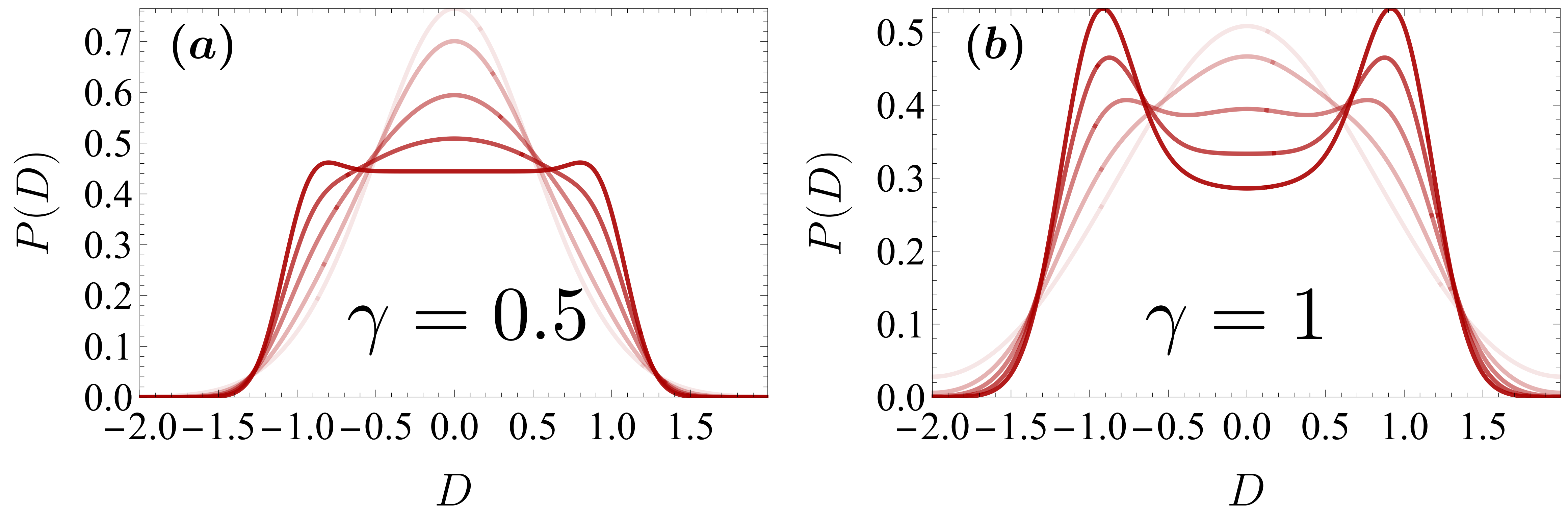
$$\langle e^{iKD} \rangle = e^{-K^2\sigma/2} \sum_m c_m \frac{(iK)^m \sigma^{m/2}}{\sqrt{m!}}$$

$$P(D) = \mathcal{F}^{-1} \langle e^{iKD} \rangle$$

Comparison with trajectories



Steady state distribution



Measurement Statistics

Statistical Moments

$$c_m = \mathrm{Tr}(M_m)$$

$$\langle D \rangle = \sqrt{\sigma} c_1$$

$$\langle D^2 \rangle = \sqrt{2}\sigma c_2 + \sigma$$

Two-Time Correlation function

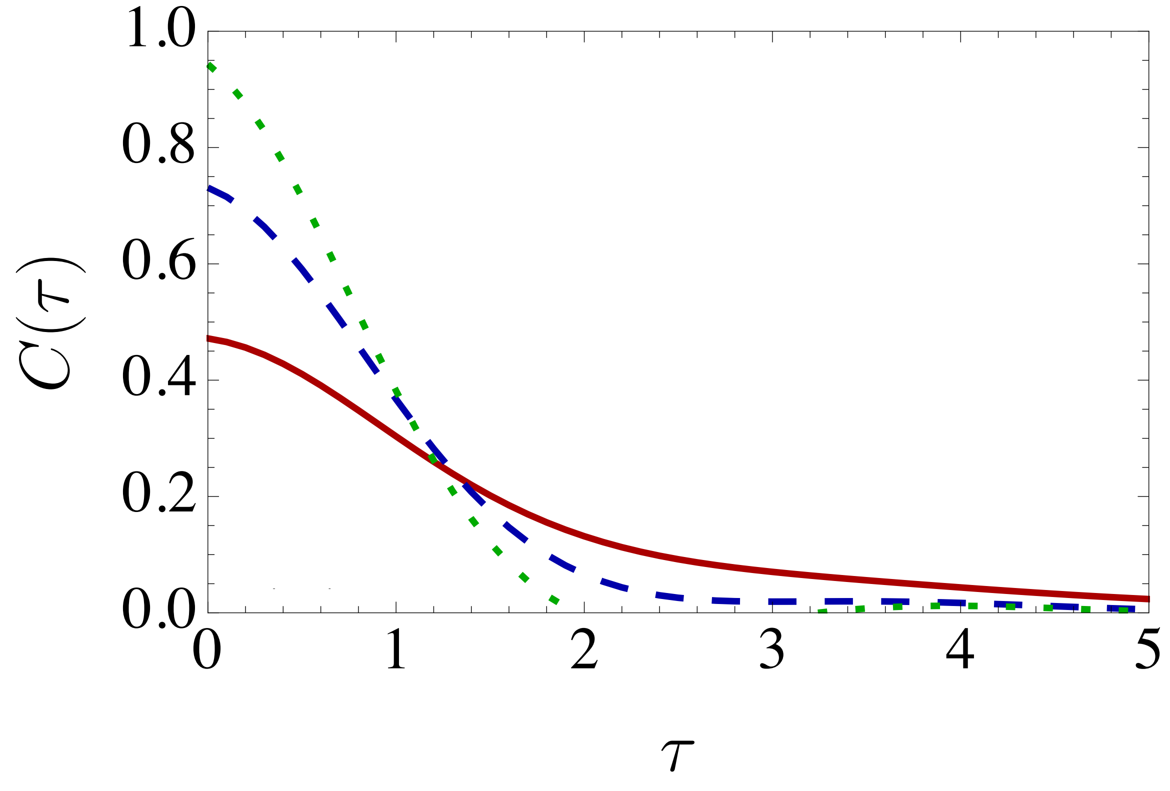
$$C(\tau) = \langle D(t+\tau) D(t) \rangle - \langle D(t) \rangle^2$$

$$= \sigma e^{-\gamma\tau} + \frac{1}{2}\sum_{j\neq 0} \frac{\gamma\left(\gamma e^{\tau\eta_j} + e^{-\gamma\tau}\eta_j\right)}{\gamma^2 - \eta_j^2} \mathrm{Tr}(A x_j)\mathrm{Tr}(y_j^\dagger \{A, M_0\})$$

Driven Qubit

$$\langle D \rangle = 0$$

$$\mathrm{Var(D)} = \frac{\gamma(\gamma + 2\lambda)}{\gamma^2 + 2\gamma\lambda + 4\Omega^2} + \frac{\gamma}{8\lambda}$$



System-Detector Correlations

Information Quantities

Mutual Information

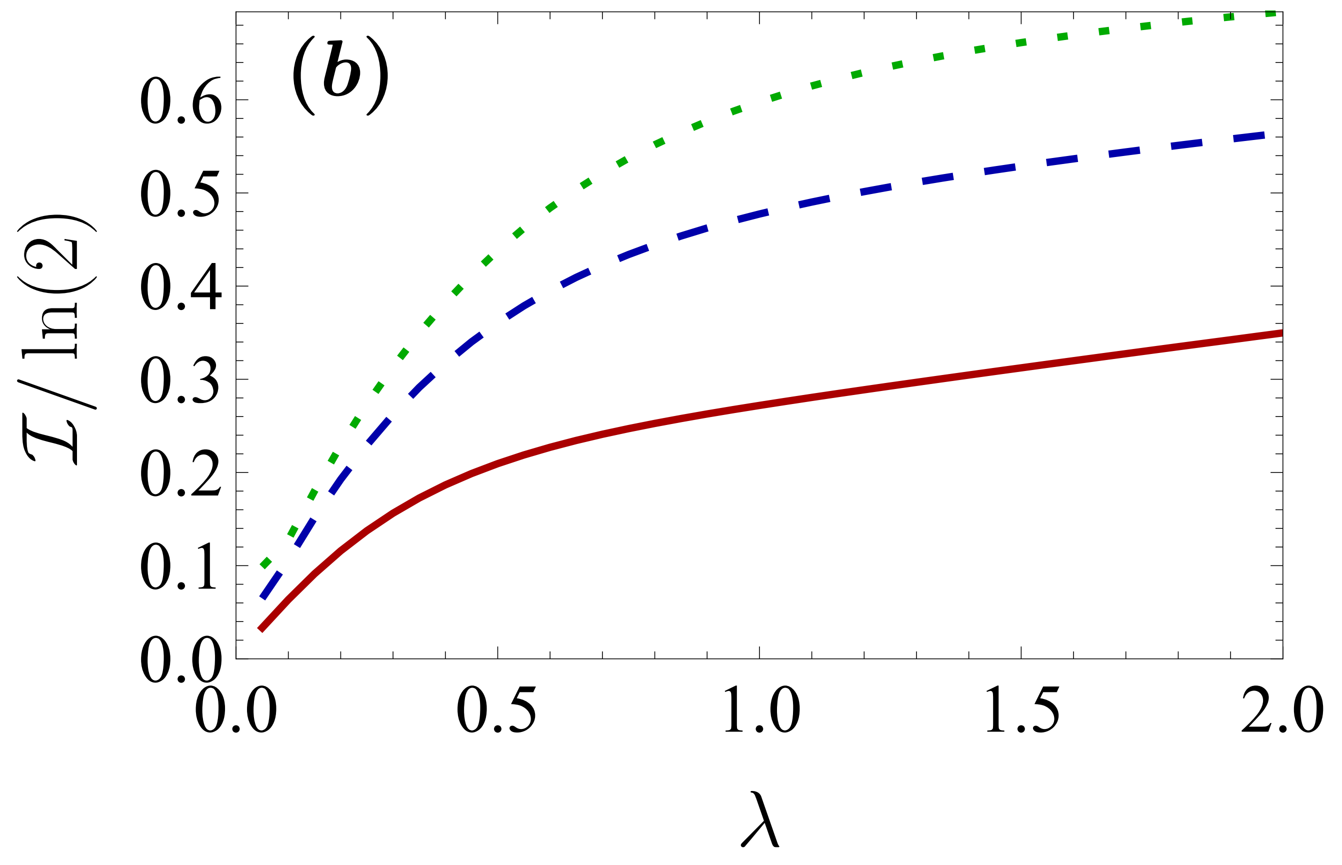
$$\mathcal{I} = S\left[\int dD \rho_t(D)\right] + S\left[\mathrm{Tr}\{\rho_t(D)\}\right] - S\left[\rho_t(D)\right]$$

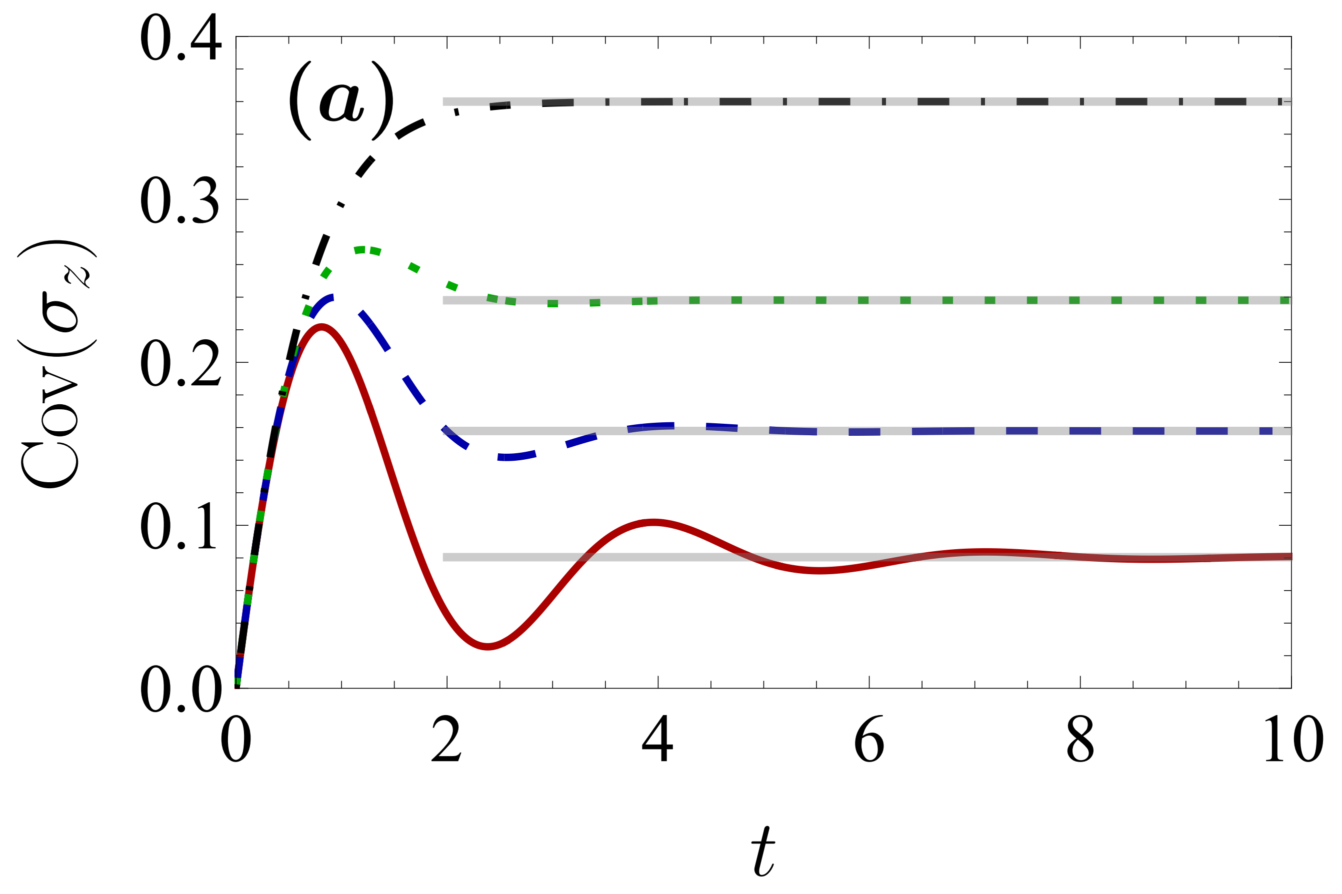
Covariance

$$\mathrm{cov(B)} = \mathrm{Tr}\int \mathrm{dD\,B\,D}\,\rho(\mathrm{D}) - \langle \mathrm{D}\rangle\langle \mathrm{B}\rangle$$

$$= \sqrt{\sigma}\left[\mathrm{Tr}(M_1 B) - \mathrm{Tr}(M_1)\mathrm{Tr}(M_0 B)\right]$$

M. M. Wolf et al., Phys. Rev. Lett. 100, 070502 (2008)



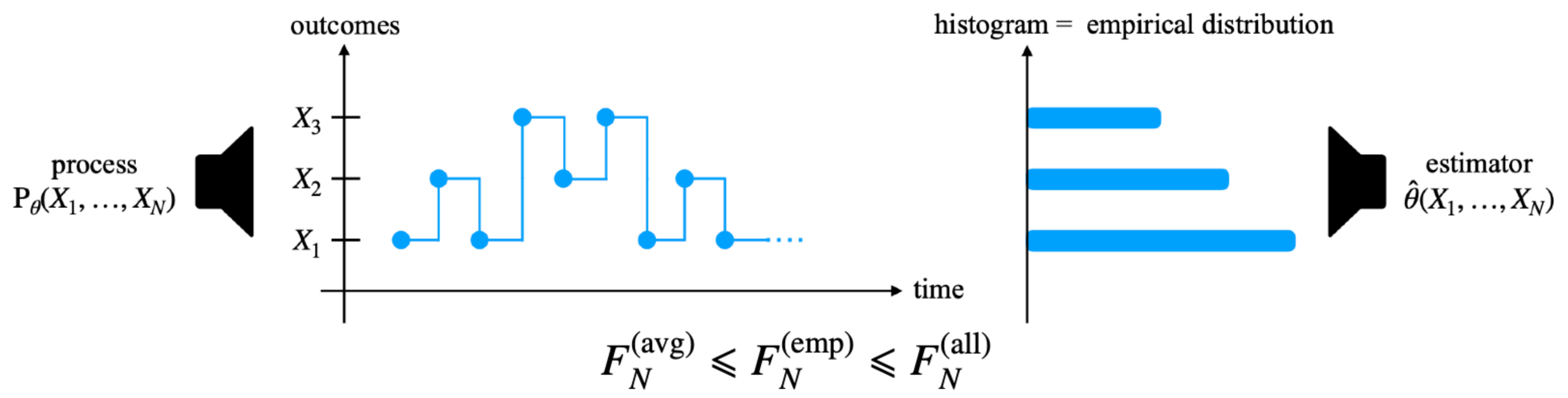


$$\mathrm{Cov}(\sigma_\mathrm{z}) = \frac{\gamma(\gamma + 2\lambda)}{\gamma^2 + 2\gamma\lambda + 4\Omega^2}$$

Quantum Metrology

Fisher Information

$$F_I = \int dD P_\mu(D)\, |\partial_\mu \ln P_\mu(D)|^2$$



J. Smiga et al., Phys. Rev. Research **5**, 033150

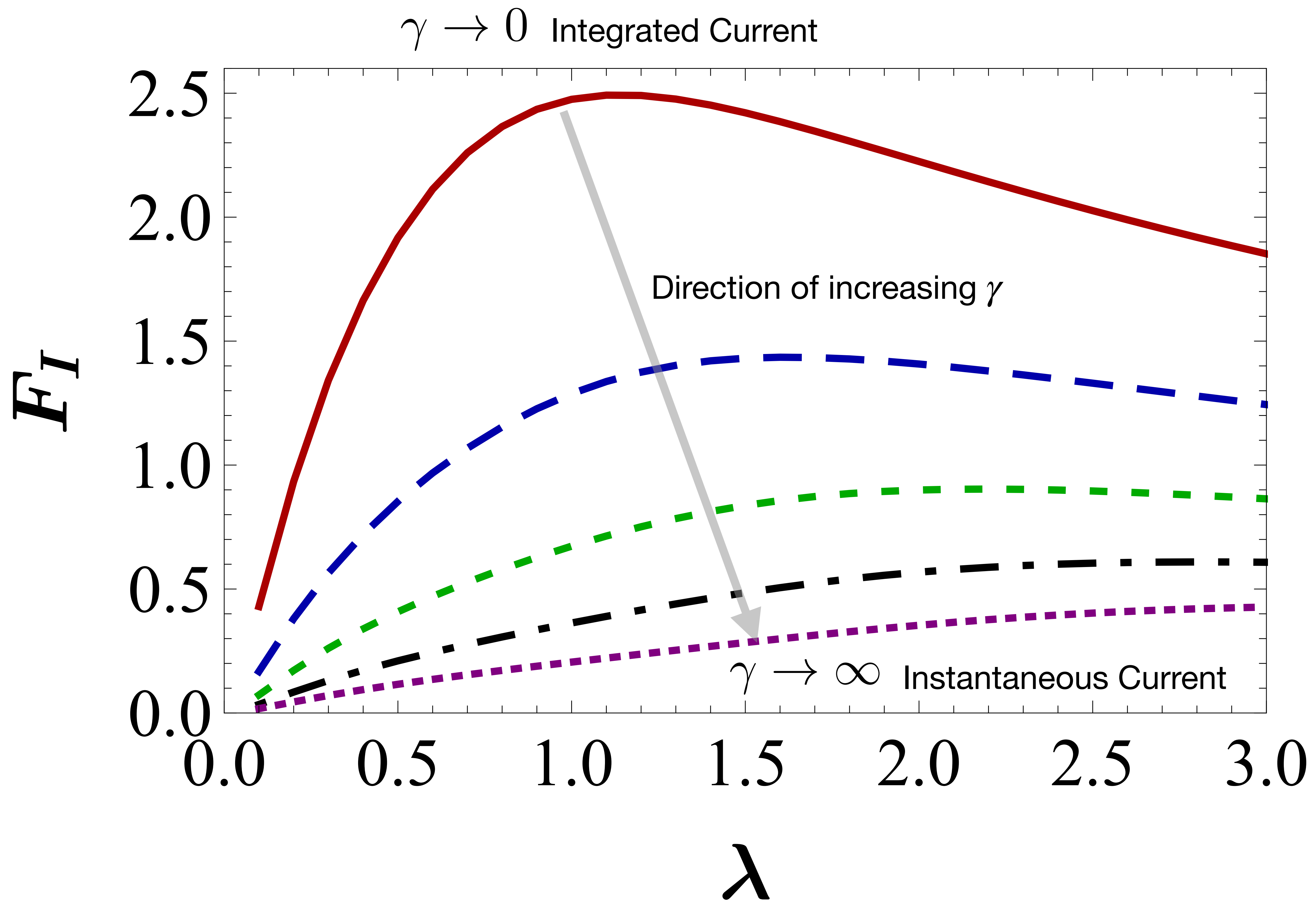
Recursion for parameter derivatives

$$\Lambda^\mu(\partial_\mu M_0^\mu) = -(\partial_\mu \mathcal{L}_0^\mu) M_0^\mu$$

$$(\Lambda^\mu - \gamma n 1_d)\,(\partial_\mu M_n^\mu) = -\,(\partial_\mu \mathcal{L}_0^\mu)\, M_n^\mu - \frac{\gamma}{2}\sqrt{\frac{n}{\sigma}}\,\{A, \partial_\mu M_{n-1}^\mu\}$$

Driven qubit $H_\mu = (\Omega + \mu)\sigma_x$

Observable $A = \sigma_z$



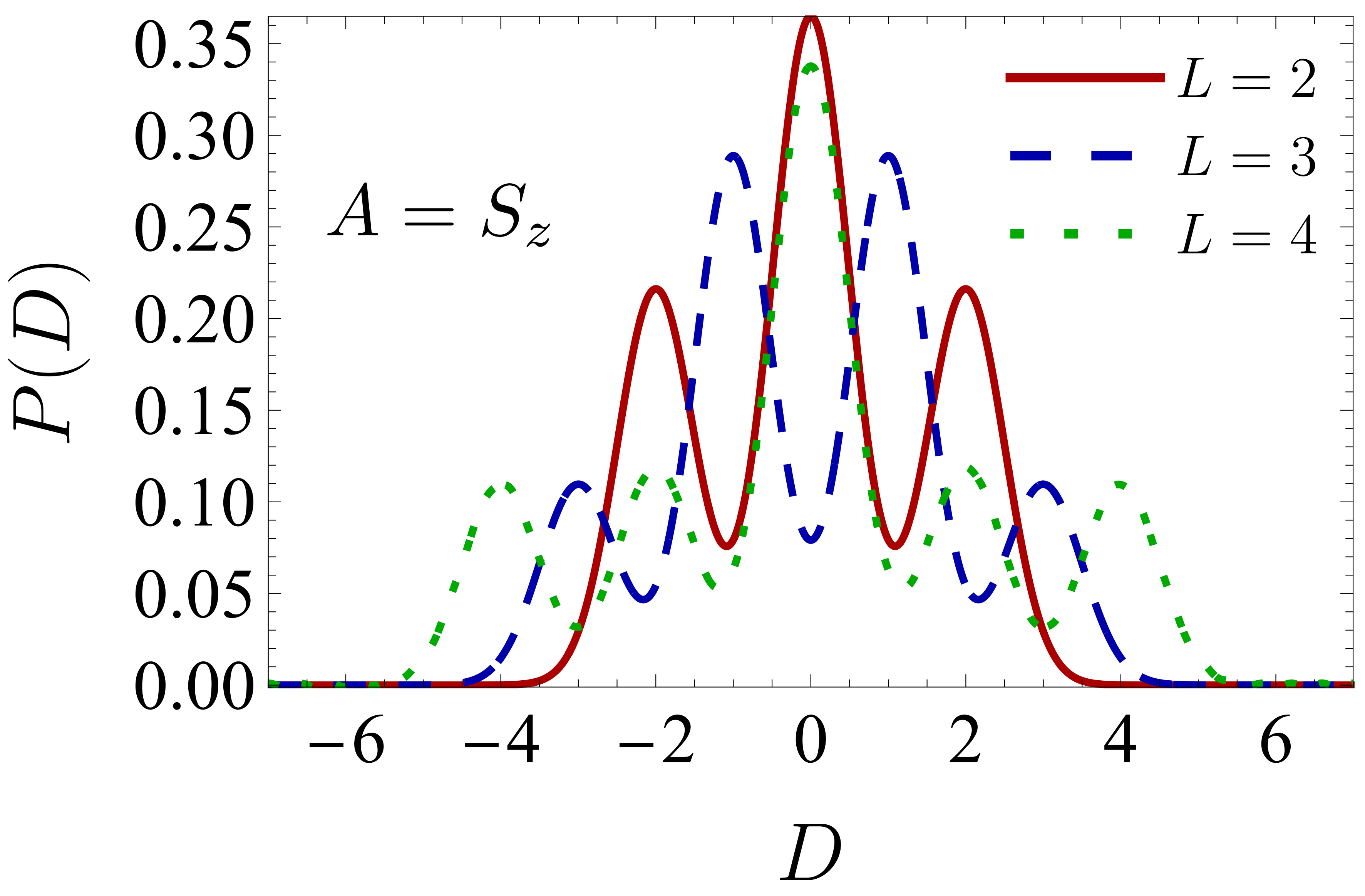
Multi-Qubit Systems

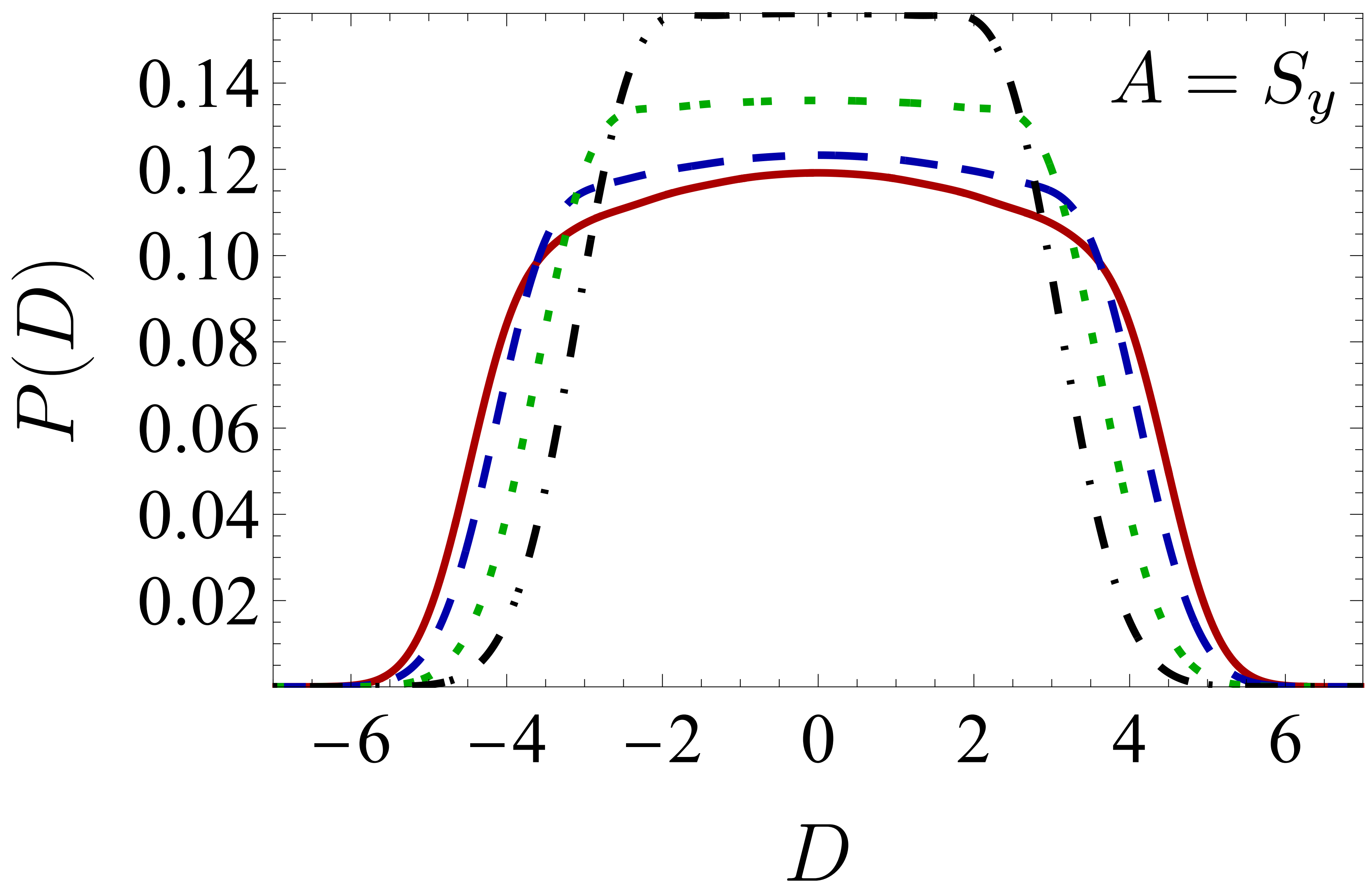
Ising model

$$H = J\sum_{j=1}^{L-1}\sigma_{j+1}^z\sigma_j^z + h\sum_{i=1}^{L}\sigma_i^x$$

Lipkin-Meskov-Glick model

$$H = -\frac{1}{L}S_x^2 + hS_z$$





Feedback Control

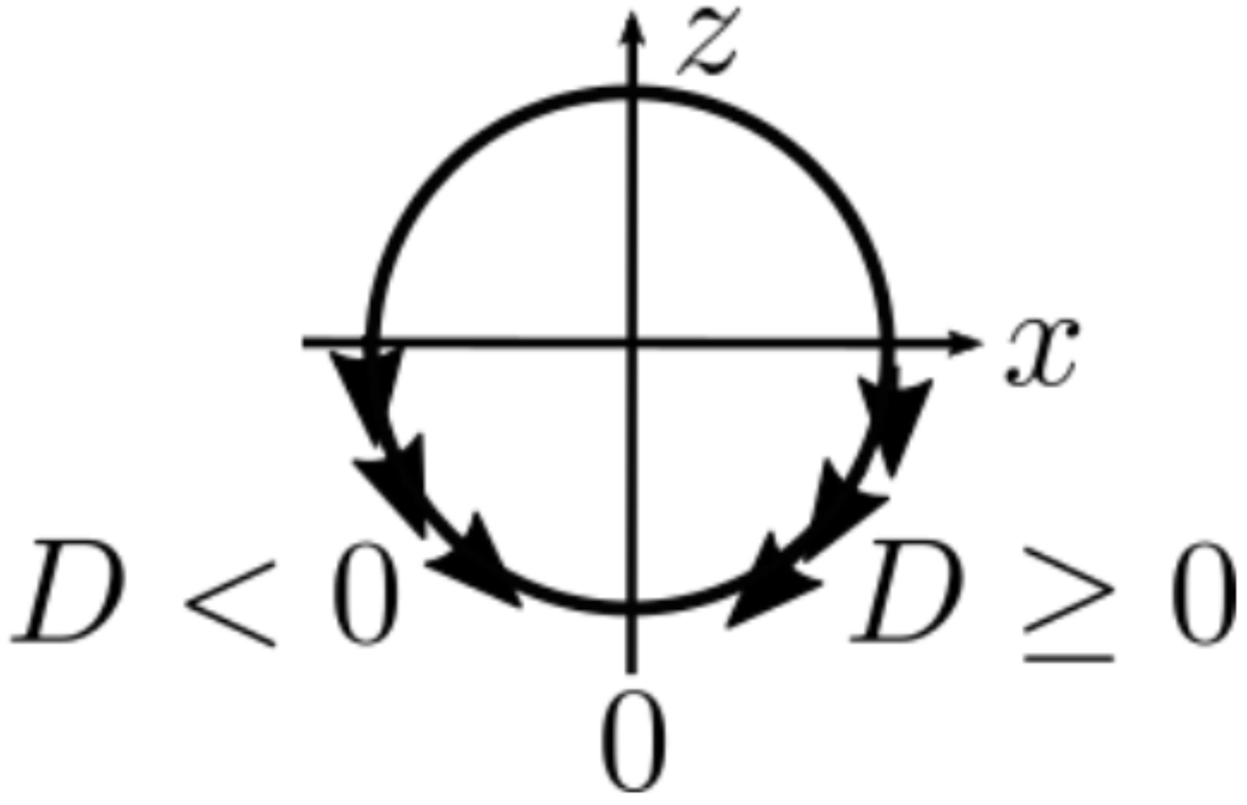
Thermal bath

$$\mathcal{L}_0 = \underbrace{\kappa n_B \mathcal{D}[\sigma_+]}_{\text{Excitation gain}} + \underbrace{\kappa(n_B+1)\mathcal{D}[\sigma_-]}_{\text{Excitation loss}}$$

Feedback correction

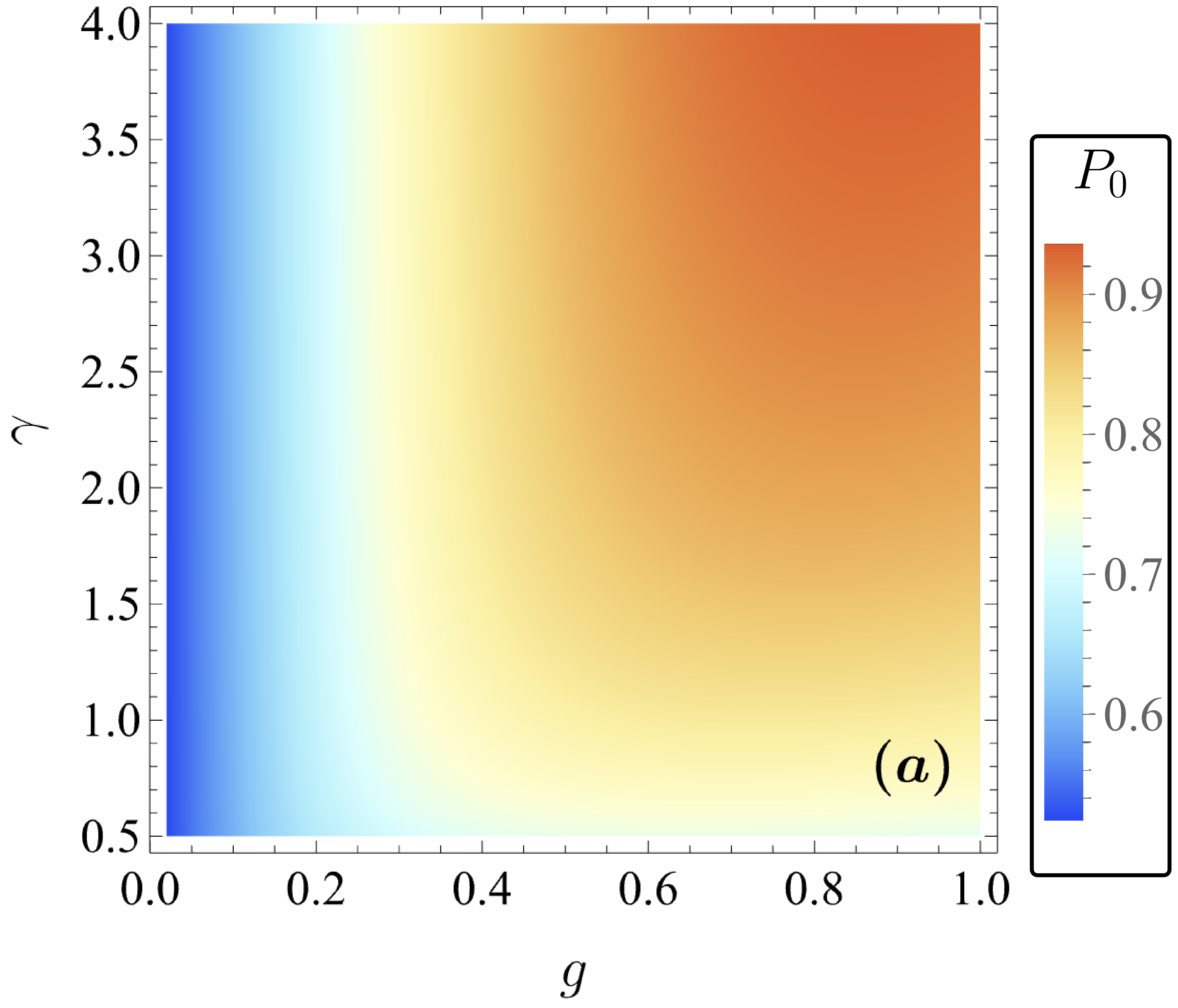
$$A = \sigma_x$$

$$\mathcal{L}_1 = -i[g\sigma_y, \cdot]$$



Ground State Probability

$$P_0 = \langle 0|M_0\,|0\rangle$$



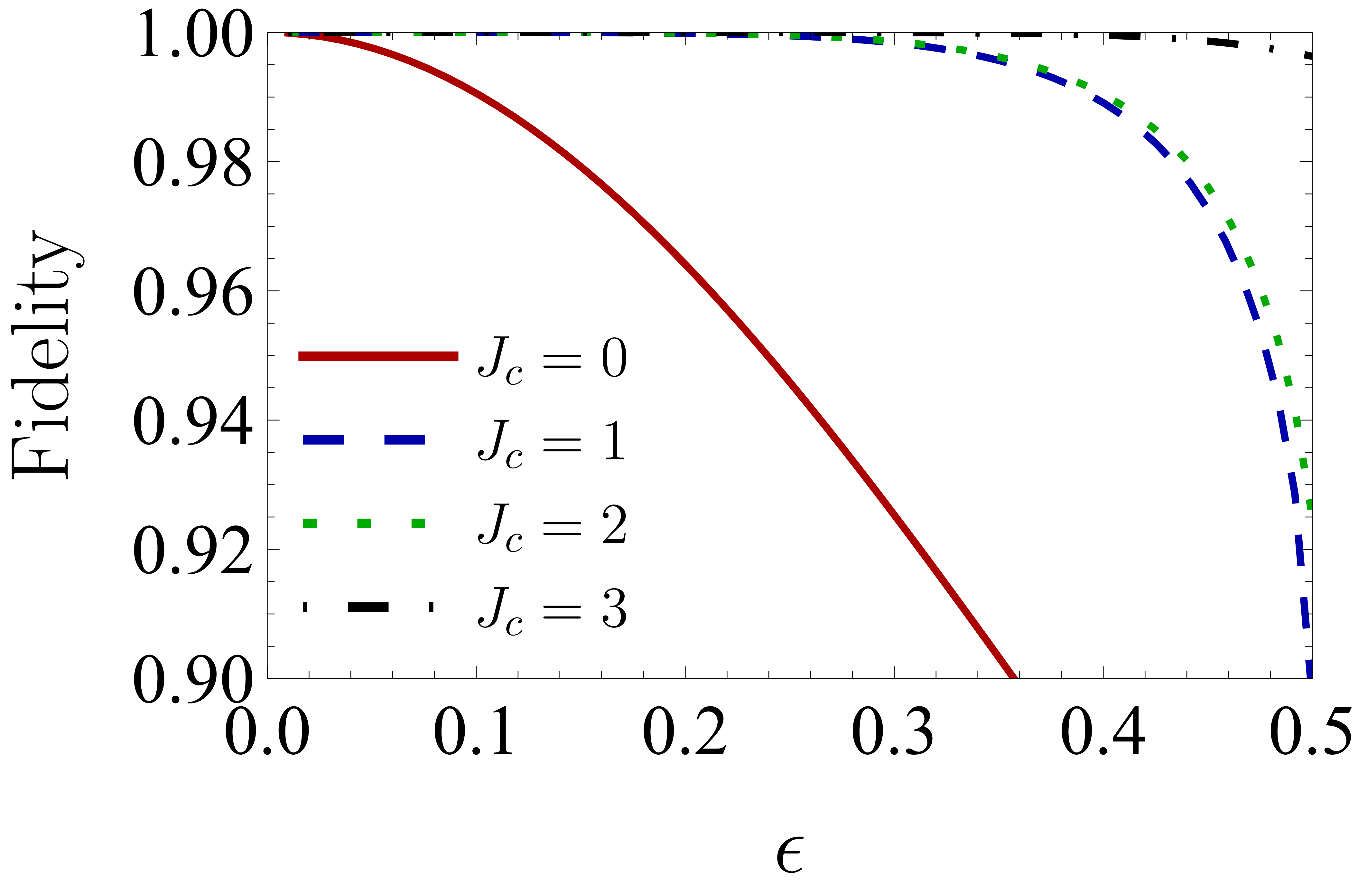
C. Elouard et. al Phys. Rev. Lett. 118, 260603 (2017)
J. Prech et. al arXiv:2505.16615 (2025)

Weak Feedback Solution

- Weak feedback $\hat{Q} = \hat{Q}_0 + \epsilon \hat{Q}_{\rm fb}$
- Perturbative solution $\vec{M} \approx \sum_{j=0}^{J_c} \epsilon^j \vec{M}^{(j)}$
- First order solution $\hat{Q}_0 \vec{M}^{(0)} = 0$
- Higher orders iteratively

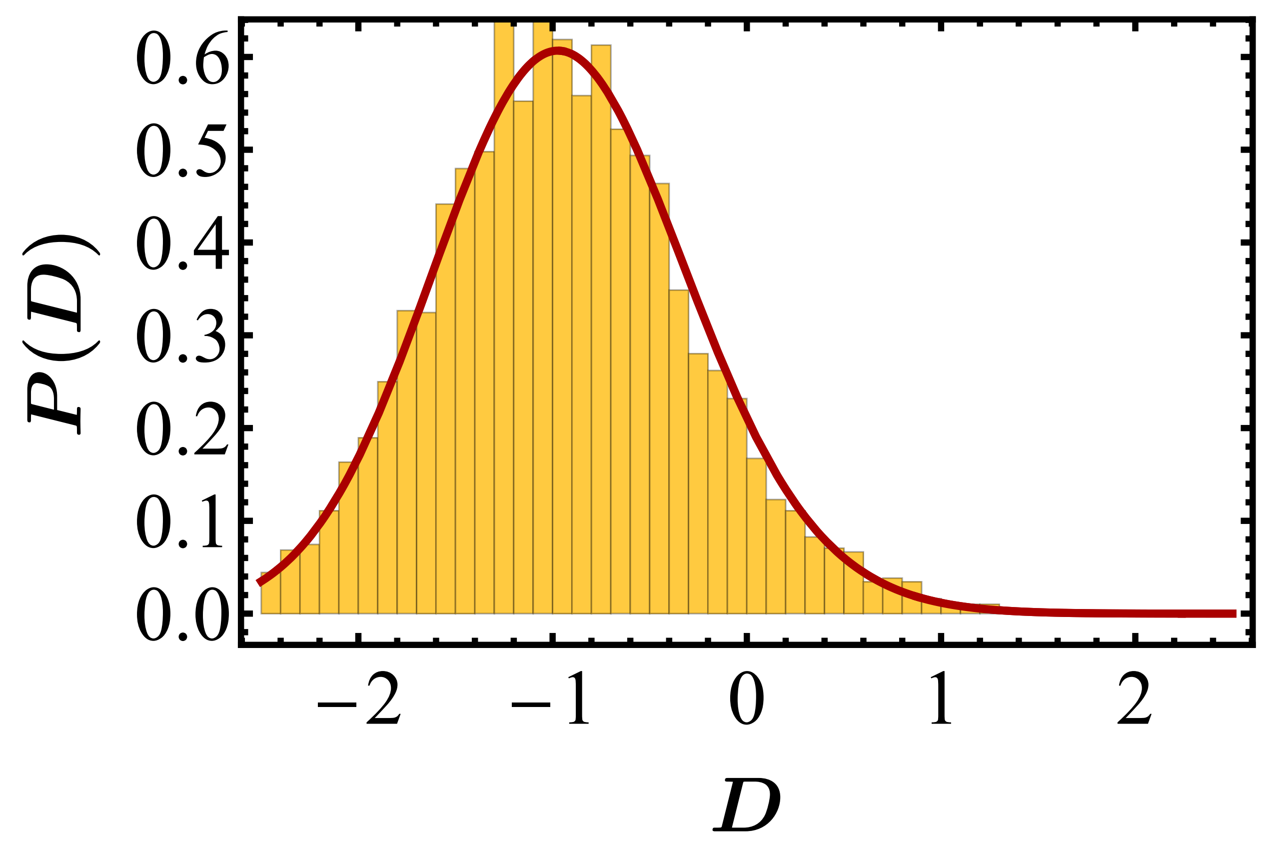
$$\hat{Q}_0 \vec{M}^{(j+1)} = -\hat{Q}_{\rm fb} \vec{M}^{(j)}$$

Comparison with exact solution



Summary

- Steady state solutions of charge resolved master equation
- Easily extract signal-detector statistics
- Perturbative solutions for feedback control



$$\hat{Q} = \begin{bmatrix} X_1 & 0 & 0 & \cdots & 0 \\ Y_1 & X_2 & 0 & \cdots & 0 \\ 0 & Y_2 & X_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & Y_{n-1} & X_n \end{bmatrix}$$

G. Landi

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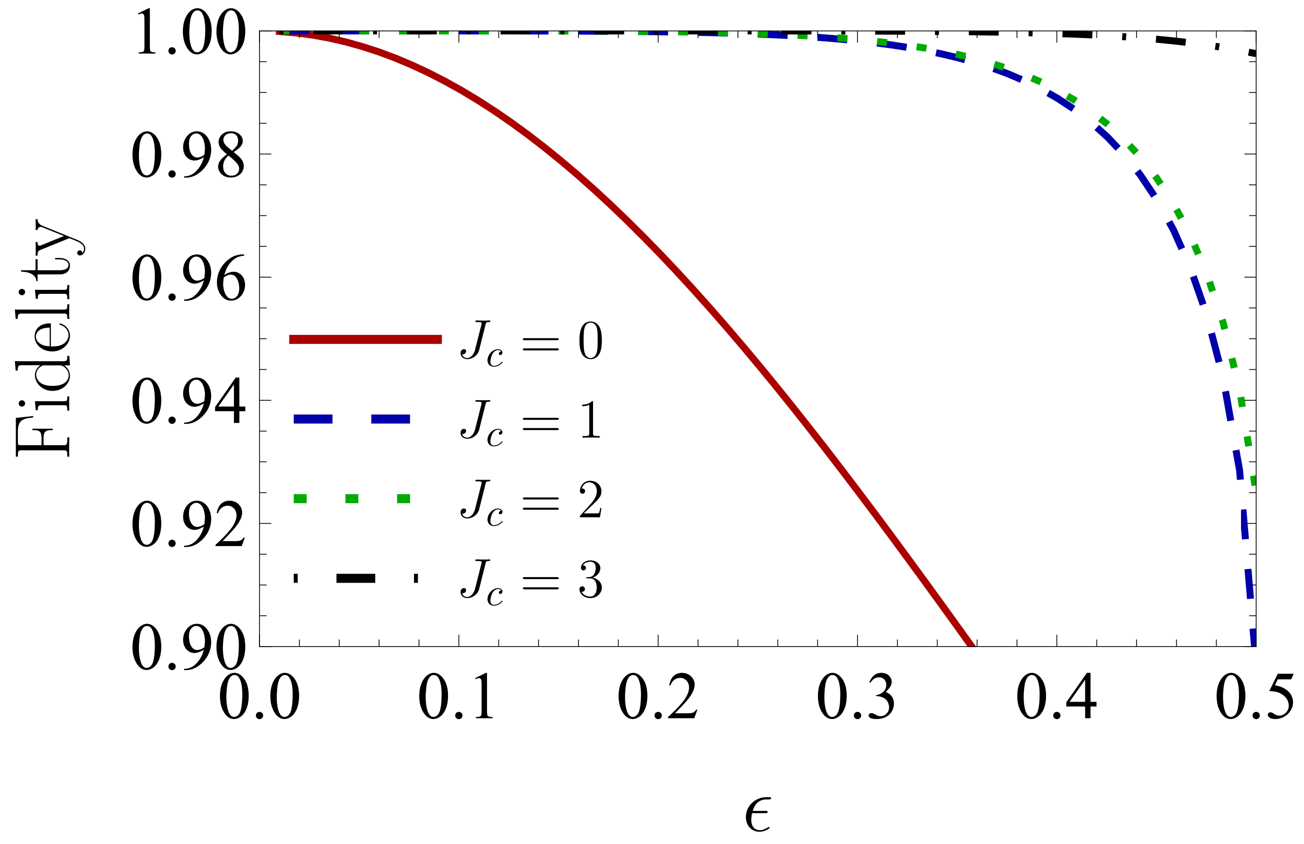

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