# Entanglement of Multi-Qubit Quantum Graph States and Quantifying Graph Properties Using Quantum Programming

**Kh. P. Gnatenko**
Ivan Franko National University of Lviv, Ukraine
SoftServe Inc.

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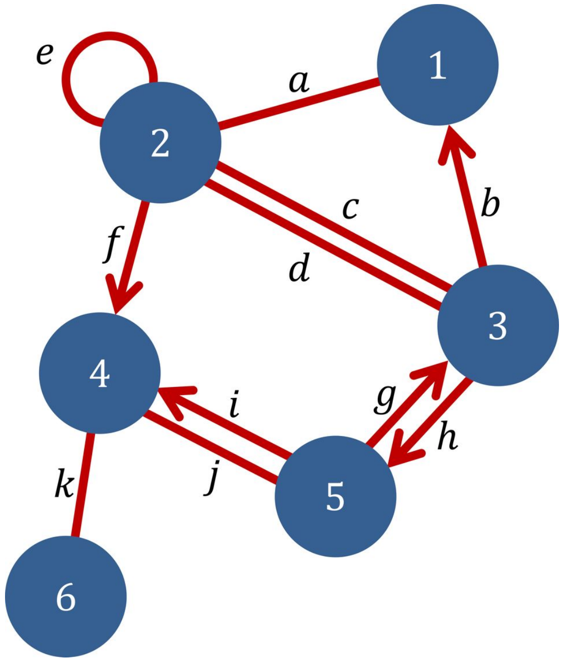
## Outline

- Definition of quantum graph states
- Entanglement in multi-qubit quantum states
- Quantum states representing weighted and directed graphs
- Relationship between geometrical properties of quantum states and structural properties of weighted graphs
- Study of properties of bi- and tripartite graphs using quantum programming

A graph is a structure consisting of vertices and edges.



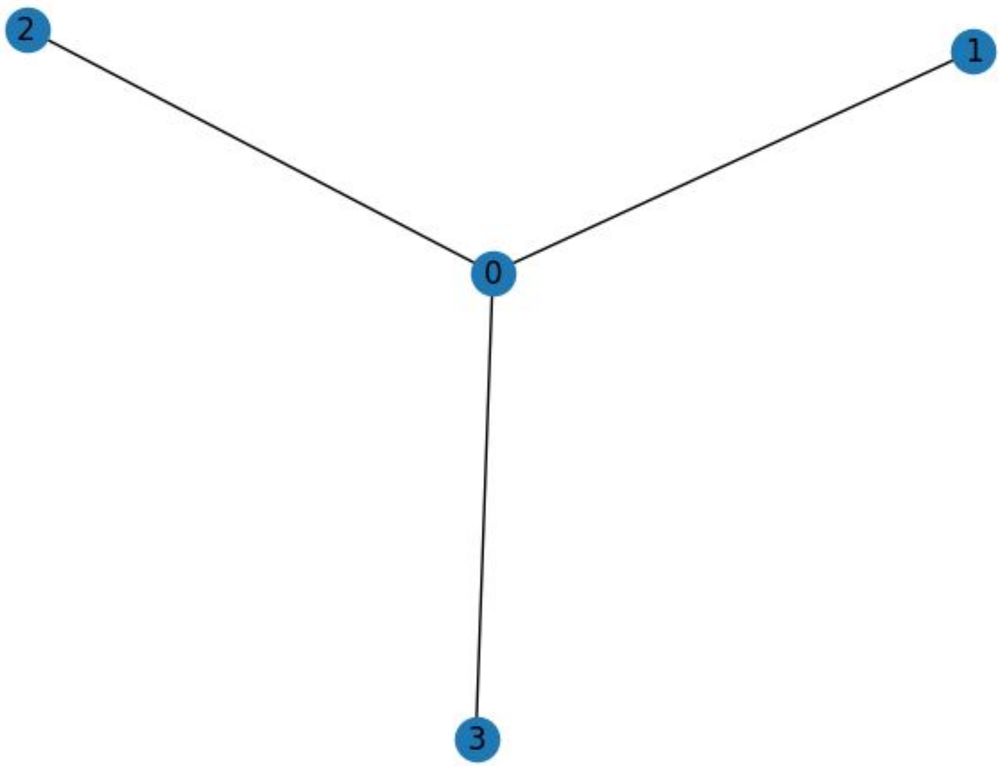

Quantum graph state

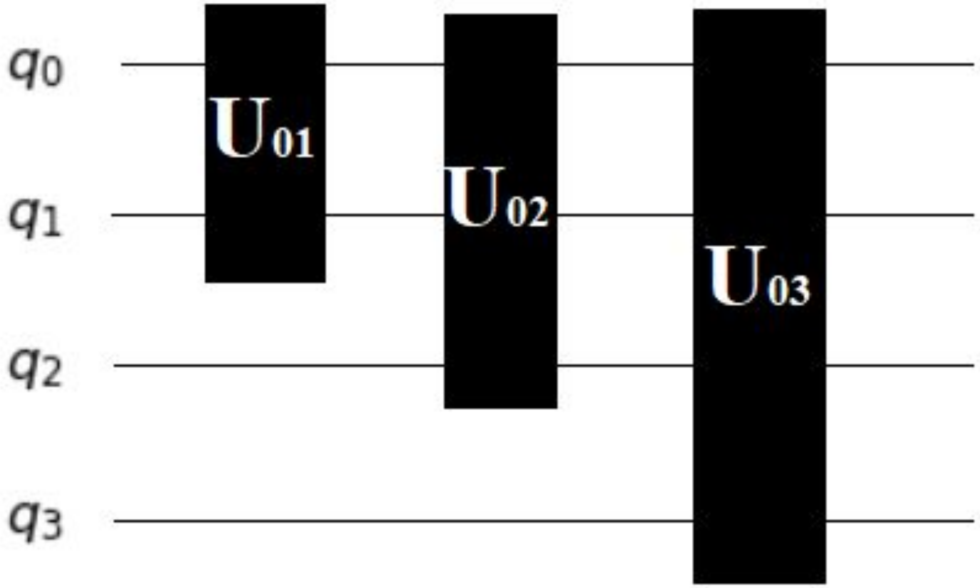
$$|\psi_G\rangle = \prod_{(i,j)\in E} U_{ij}|\psi_{init}\rangle,$$

$U_{ij}$ – two-qubit operator,
$|\psi_{init}\rangle$ – initial state.

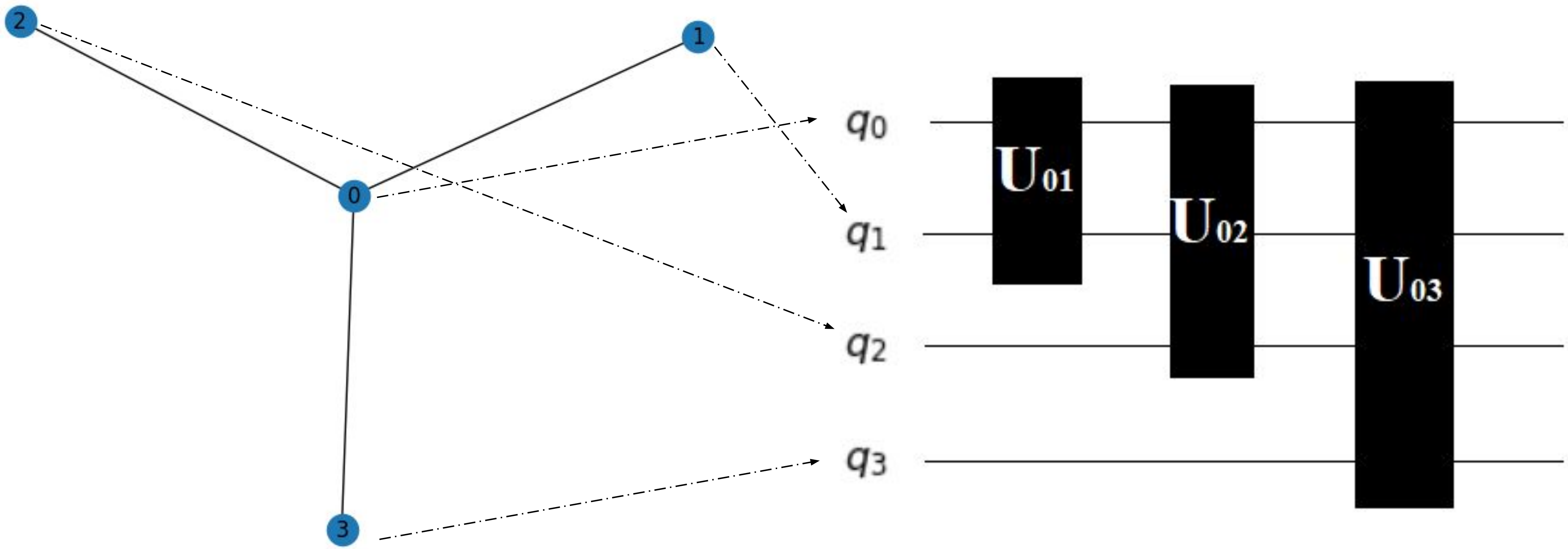
Quantum graph state – quantum state that can be represented with a graph

# Quantum graph states

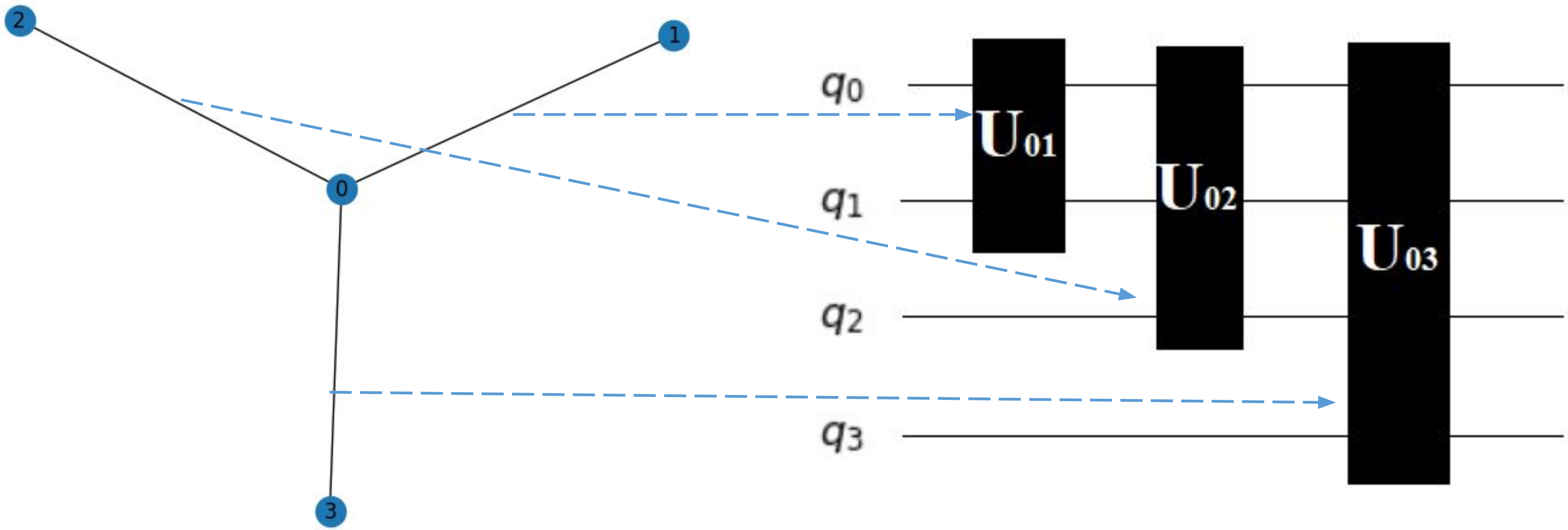


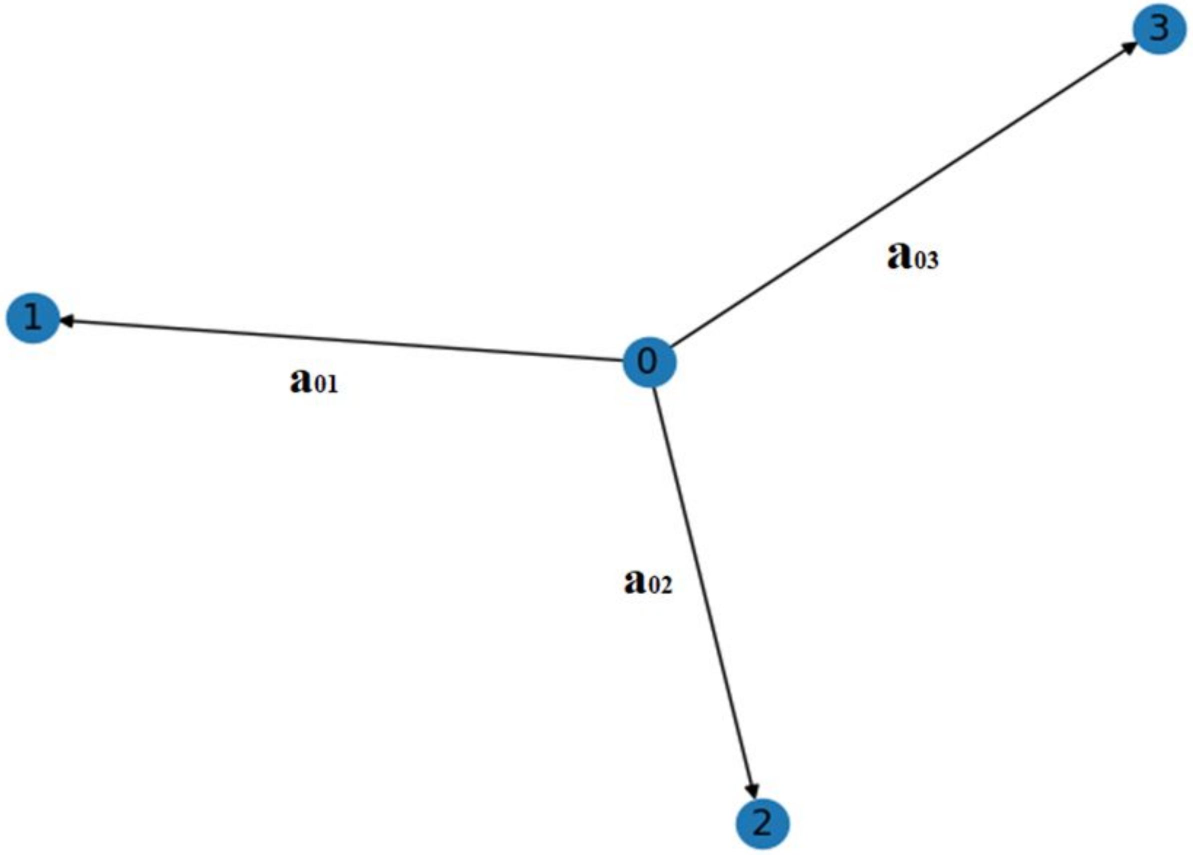
# Quantum graph states

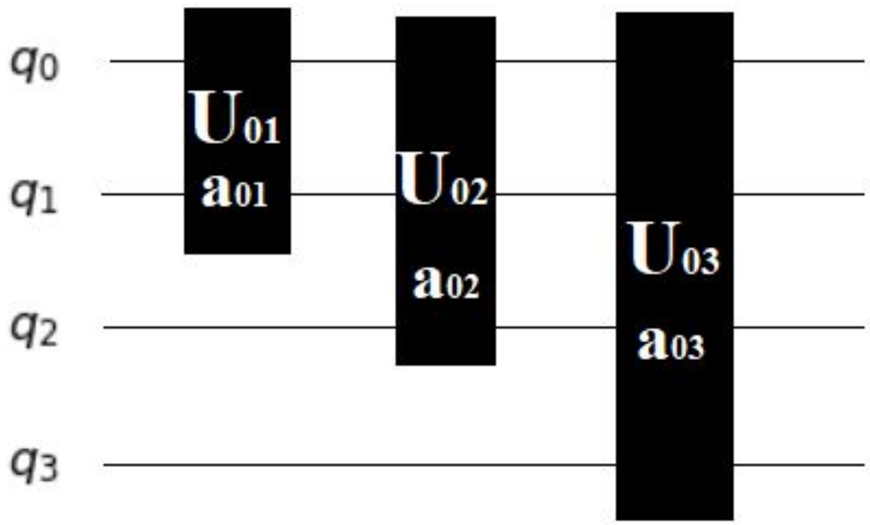


# Quantum graph states



# Quantum graph states corresponding to weighted and directed graphs



# Quantum graph states

In the literature, considerable attention has been devoted to the study of quantum graph states.

$$|\psi_G\rangle = \prod_{(i,j)\in E} CZ_{ij}(\phi)\,|+\rangle^{\otimes V}.$$

[G. J. Mooney, Ch. D. Hill, L. C. L. Hollenberg, Sci. Rep. 9, 13465 (2019).]

In paper [Kh. P. Gnatenko, N. A. Susulovska EPL 136(4), 40003 (2021)] entanglement of multiqubit states has been examined

$$|\psi_G(\phi,\alpha,\theta)\rangle = \prod_{(i,j)\in E} CP_{ij}(\phi)\,|\psi(\alpha,\theta)\rangle^{\otimes V},$$

$$|\psi(\alpha,\theta)\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\alpha}\sin\frac{\theta}{2}|1\rangle,$$

$$\boxed{E_l = \frac{1}{2} - \frac{1}{2}\sqrt{\sin^2\theta\left(\cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2}\cos^2\theta\right)^{n_l} + \cos^2\theta}.}$$

$n_l$ - degree of vertex l in a graph.

# Geometric measure of entanglement

Geometric measure of entanglement is defined as

$$E(|\psi\rangle) = \min_{|\psi_s\rangle}(1 - |\langle\psi|\psi_s\rangle|^2).$$

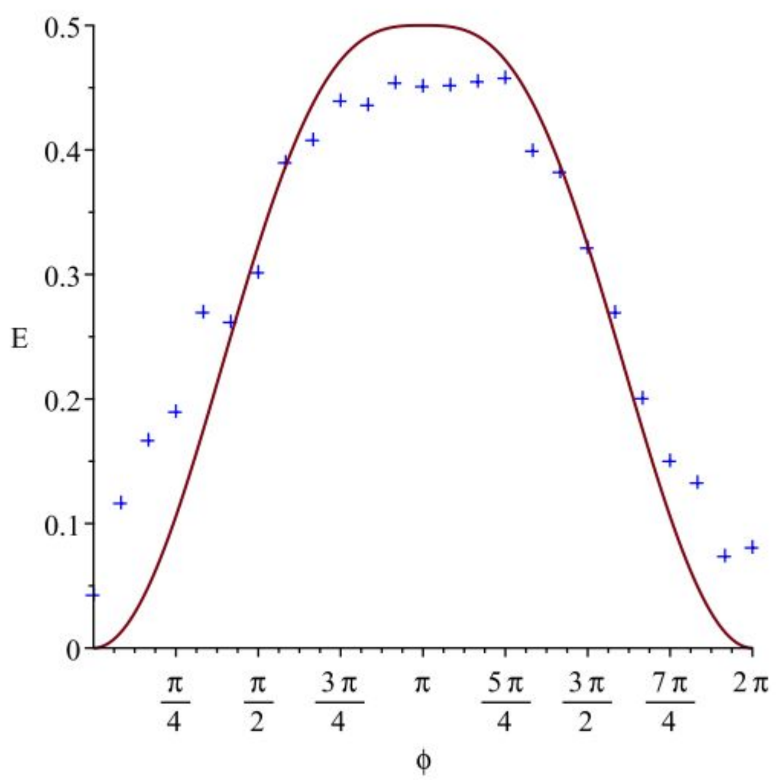
$|\psi\rangle$ is the entangled state, $|\psi_s\rangle$ is a set of non-entangled states

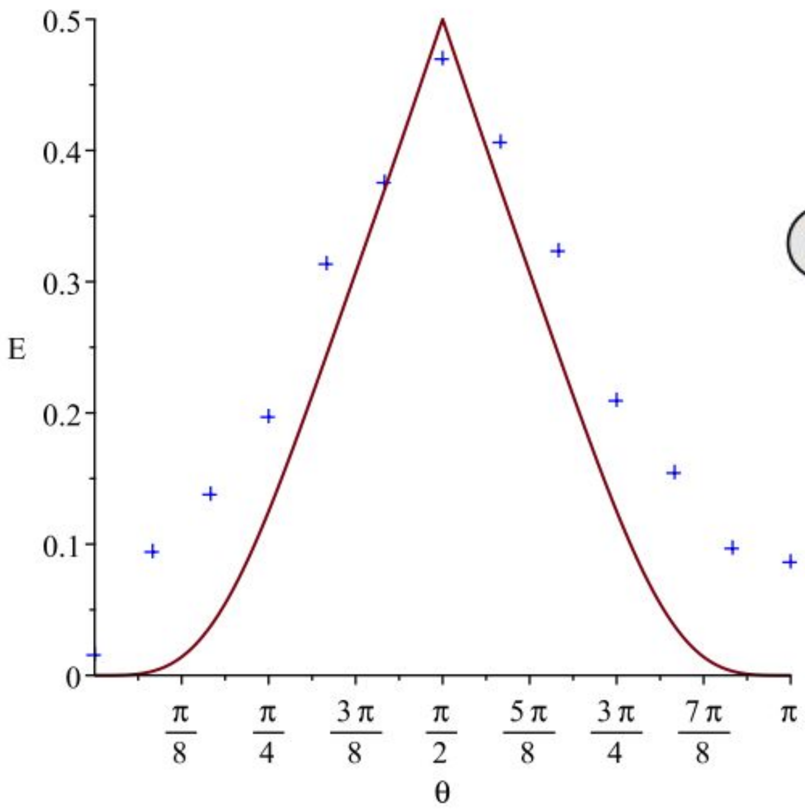
[A. Shimony, Ann. N.Y. Acad. Sci. 755, 675 (1995)].

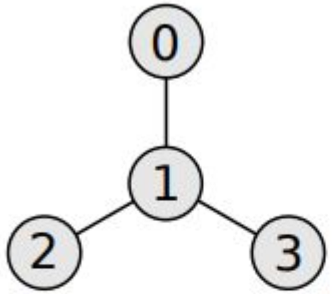
$$E(|\psi\rangle) = \frac{1}{2}(1 - |\langle\boldsymbol{\sigma}\rangle|), \quad \langle|\boldsymbol{\sigma}|\rangle = \sqrt{\langle\boldsymbol{\sigma}\rangle}.$$

[A. M. Frydryszak, M. I. Samar, V. M. Tkachuk, Eur. Phys. J. D 71, 233 (2017)]

# Results of quantum calculations of the entanglement (ibmq_athens)




[Kh. P. Gnatenko, N. A. Susulovska EPL 136(4), 40003 (2021)]

# Entanglement of quantum states representing weighted and directed graphs

Quantum graph states

$$|\psi_G\rangle = \prod_{(i,j)\in A} RXX_{ij}(\phi_{ij})\,|\psi_0\rangle\,. \qquad |\psi_0\rangle = \prod_{k\in V} |\psi(\alpha_k,\theta_k)\rangle_k\,,$$

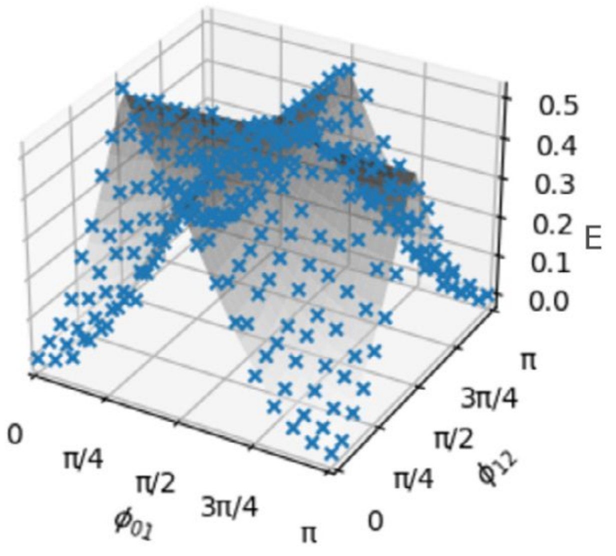
$$|\psi(\alpha_k,\theta_k)\rangle_k = \cos\frac{\theta_k}{2}\,|0\rangle_k + e^{i\alpha_k}\sin\frac{\theta_k}{2}\,|1\rangle_k = e^{i\frac{\alpha_k}{2}} RZ(\alpha_k) RY(\theta_k)\,|0\rangle_k\,.$$

Entanglement of quantum graph states

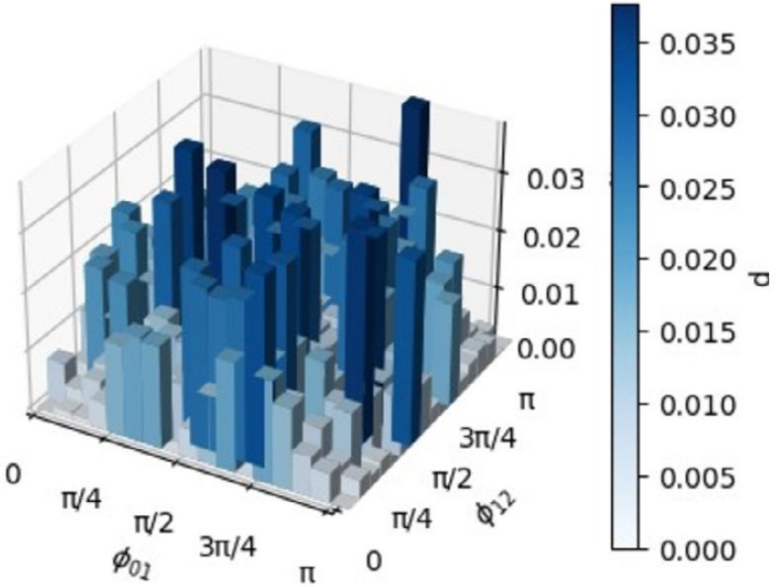
$$E_k(|\psi_G\rangle) = \frac{1}{2} - \frac{1}{2}\Big[\cos^2\alpha\sin^2\theta + \left(\cos^2\theta + \sin^2\alpha\sin^2\theta\right)\times$$
$$\times\left(\cos^2\phi_k^{(in)} + \sin^2\phi_k^{(in)}\cos^2\alpha\sin^2\theta\right)^{n_k^{(in)}}\times$$
$$\left(\cos^2\phi_k^{(out)} + \sin^2\phi_k^{(out)}\cos^2\alpha\sin^2\theta\right)^{n_k^{(out)}}\Big]^{\frac{1}{2}}.$$

Kh. P. Gnatenko Phys. Lett. A. Vol. 521. Art. 129815. 5 p (2024)

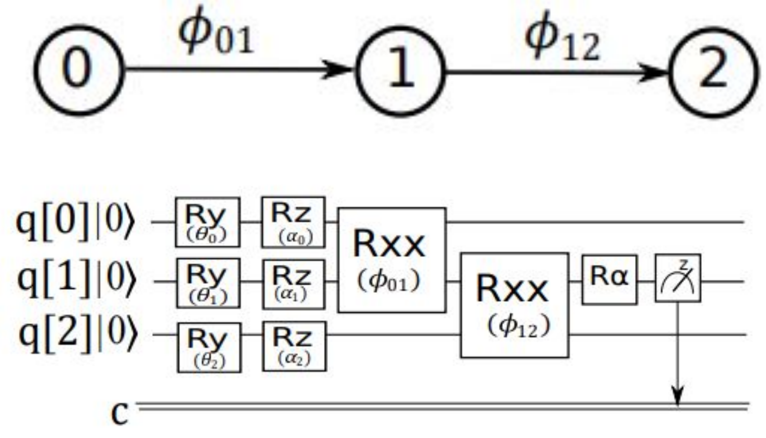
# Entanglement of quantum states representing weighted and directed graphs




(a)




(b)




Quantum-computed entanglement (cross markers) and analytical results (surface).
The 3D histograms show the difference between the quantum computation results and the theoretical results.

$$| G\rangle = \prod_{(a,b)\in E} RXX_{ab}(\phi_{ab}) \prod_{i} RZ_i(\alpha_i) RY_i(\theta_i) \mid 0\rangle^{\otimes n},$$

Kh. P. Gnatenko Phys. Lett. A.  Vol. 521. Art. 129815. 5 p (2024)

# Quantum computing of entanglement of quantum states

- The entanglement of a single qubit (spin) with other qubits (spins) in evolving quantum states of spin systems and in multi-qubit quantum graph states with arbitrary structure has been determined.
- A relationship between entanglement and the node degree, as well as the weights of edges incident to the corresponding node in the graph, has been established.
- The entanglement of quantum states has been computed using quantum computers and quantum simulators developed by IBM.

# Geometric characteristics of evolutionary weighted graph states

Let us consider a system described by the Ising model with Hamiltonian

$$H_I = \frac{1}{2}\sum_{i,j} J_{ij}\sigma_i^z\sigma_j^z,$$

where $\sigma_i^z$ is the Pauli matrix that corresponds to spin $i$. Constants $J_{ij}$ are interaction couplings. The evolutionary state of the system

$$|\psi(t)\rangle = \exp\left(-\frac{iH_It}{\hbar}\right)|\psi_0\rangle$$

can be considered as a quantum graph state

$$|\psi_G\rangle = \prod_{(i,j)\in E} RZZ_{ij}(\phi_{ij})\,|\psi_0\rangle\,. \quad \phi_{ij} = 2J_{ij}t/\hbar.$$

# Curvature and torsion

The curvature and torsion are related with the fluctuations of energy

$$\frac{\gamma^2}{R^2} = \frac{\langle(\Delta H)^4\rangle - \langle(\Delta H)^2\rangle^2}{\langle(\Delta H)^2\rangle^2}, \qquad \bar{\tau} = \frac{\langle(\Delta H)^4\rangle - \langle(\Delta H)^2\rangle^2}{\langle(\Delta H)^2\rangle^2} - \frac{\langle(\Delta H)^3\rangle^2}{\langle(\Delta H)^2\rangle^3}$$

$$\frac{\gamma^2}{R^2} = \frac{96 S_4 + 2(\sum_i n_i^{(2)})^2 - 4\sum_i n_i^{(4)}}{(\sum_i n_i^{(2)})^2}, \qquad \bar{\tau} = \frac{96 S_4 + 2(\sum_i n_i^{(2)})^2 - 4\sum_i n_i^{(4)}}{(\sum_i n_i^{(2)})^2} - \frac{288 S_3^2}{(\sum_i n_i^{(2)})^3}$$

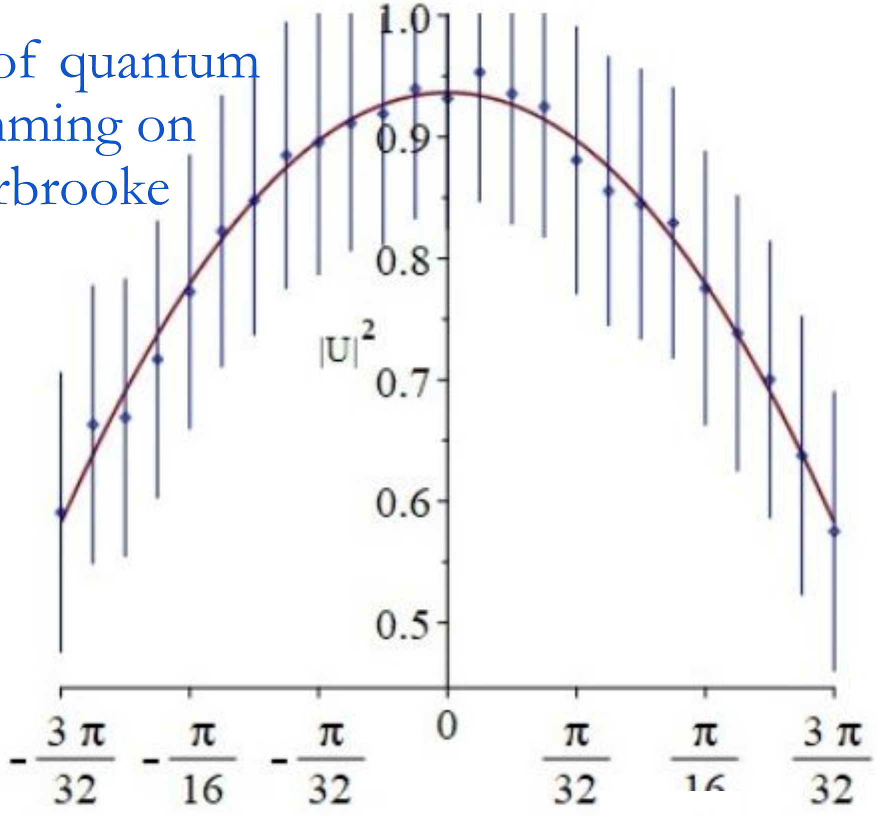
where $n_i^{(4)} = \sum_j J_{ij}^4$ is the weighted degree of $i$-th node in graph $G^{(4)}(V, E)$, constructed raising to the fourth the weights of corresponding edges in $G(V, E)$ (the elements of adjancecy matrix of $G^{(4)}(V, E)$ are $J_{ij}^4$).

$$S_3 = \frac{1}{6}\sum_{i,j,k} J_{ij} J_{jk} J_{ki} \qquad S_4 = \frac{1}{8}\sum_{i,j,k,l} \sum_{\substack{i\neq k,\\ j\neq l}} J_{ij} J_{jk} J_{kl} J_{li}$$

# Studies of structural properties of weighted and directed graphs with quantum programming

$$H_I = \frac{1}{2}\sum_{i,j} J_{ij}\sigma_i^z\sigma_j^z, \qquad |\langle U\rangle|^2 = |\langle\psi_0|\exp\left(\frac{-iH_I t}{\hbar}\right)|\psi_0\rangle|^2 = 1 - \frac{\langle\Delta H_I^2\rangle}{\hbar^2}t^2.$$

Results of quantum programming on ibm sherbrooke



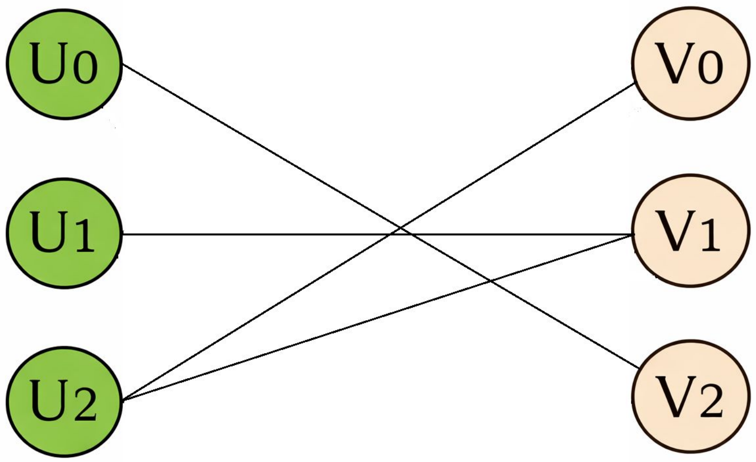

Results of quantum computing for $|\langle U\rangle|^2$ for different $\phi$.

- Quantum algorithms have been developed for studying the structural properties of weighted graphs. The algorithms are based on a discovered relationship between the geometric properties of quantum graph states and the structural properties of the corresponding graphs.
- These algorithms open the possibility of achieving quantum advantage in solving problems related to determining the products of the weights of edges that form closed cycles of lengths 3 and 4 in graphs.

Gnatenko Kh. P., Eur. Phys J Plus Vol. 140, 241 (2025).

# Quantum states representing bipartite graphs

**A bipartite graph G(U,V,E)** is a type of graph in which the set of vertices can be divided into two separate and disjoint groups U and V, such that every edge connects a vertex from set U to a vertex from set V.




$$|\psi\rangle = \prod_{(u,v)\in E} CNOT_{uv}\,|\psi_{init}\rangle\,. \qquad CNOT_{uv} = e^{i\frac{\pi}{4}(1-\sigma_u^z)(1-\sigma_v^x)}$$

$$|\psi_{init}\rangle = |\psi_{init}^{(U)}\rangle\,|\psi_{init}^{(V)}\rangle\,, \qquad |\psi_{init}^{(A)}\rangle = \prod_{k\in A}\left(\cos\frac{\theta_k^{(A)}}{2}|0\rangle_k + e^{i\phi_k^{(A)}}\sin\frac{\theta_k^{(A)}}{2}|1\rangle_k\right),$$

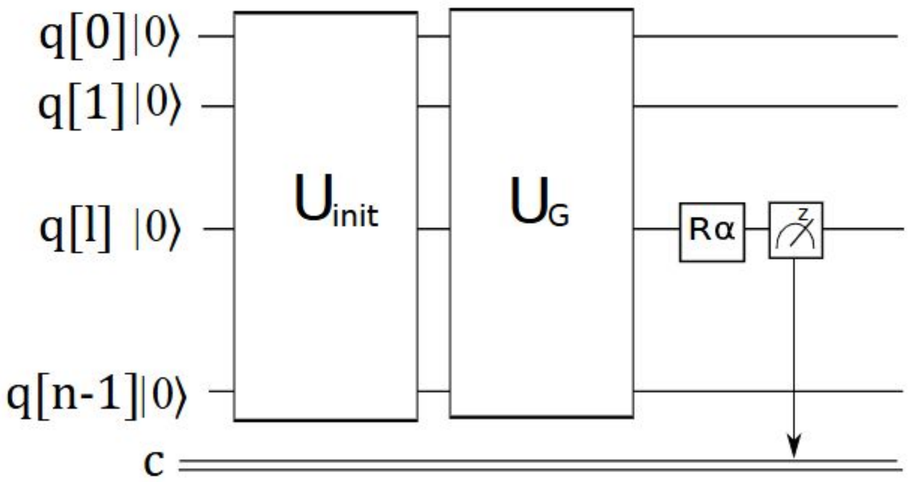
Kh. P. Gnatenko, Phys. Lett. A 566, 131191 [7 p.] (2026)

# Entanglement distance

The entanglement distance of qubit l (spin l) with other qubits in state $|\psi\rangle$ reads

$$E_l^{ED}(|\psi\rangle) = 1 - \sum_{j=x,y,z} \langle\psi|\,\sigma_l^j\,|\psi\rangle^2 \,,$$

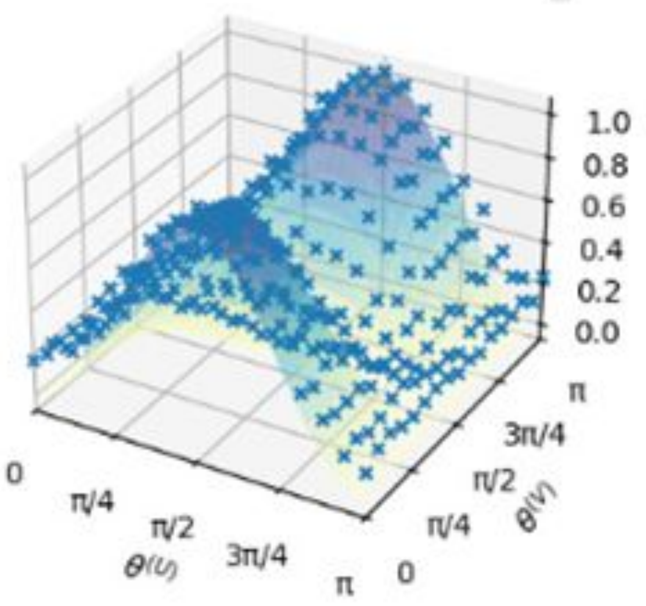
D. Cocchiarella et al Phys. Rev. A, vol. 101, Art. 042129, 2020.
A.Vesperini et al Front. Phys. vol. 19, Art. 51204, 2024.



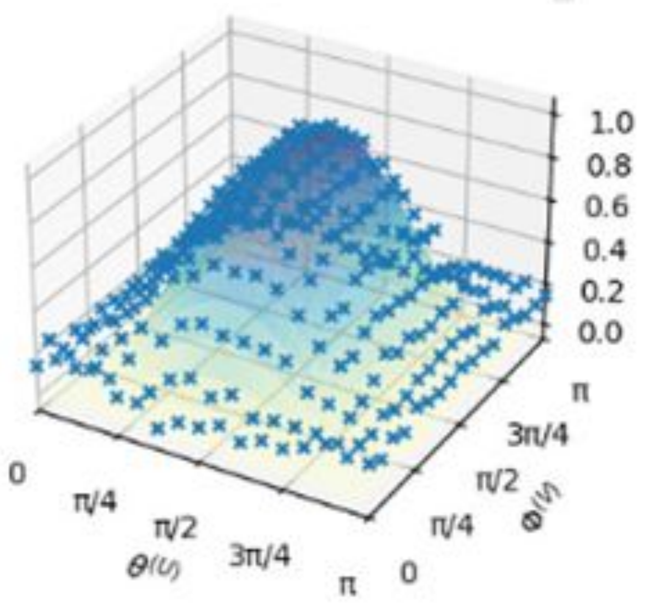
# Results of analytical studies of the entanglement and quantum programming

$$E_v^{ED}(|\psi\rangle) = 1 - \cos^2\phi_v^{(V)}\sin^2\theta_v^{(V)} - (\sin^2\phi_v^{(V)}\sin^2\theta_v^{(V)} + \cos^2\theta_v^{(V)})(\cos\theta^{(U)})^{2n_v}.$$

$$E_u^{ED}(|\psi\rangle) = \sin^2\theta_u^{(U)}(1 - (\cos\phi^{(V)}\sin\theta^{(V)})^{2n_u}).$$




(a)




(b)

Kh. P. Gnatenko, Phys. Lett. A 566, 131191 [7 p.] (2026)

# Quantum correlators and number of vertices with odd and even degrees

$$\langle\psi| \prod_{u\in U} \prod_{v\in V} \sigma_u^x \sigma_v^x |\psi\rangle = (\cos\phi^{(U)} \sin\theta^{(U)})^{|U|} (\cos\phi^{(V)} \sin\theta^{(V)})^{|V_{enen}|}.$$

$$\langle\psi| \prod_{u\in U} \prod_{v\in V} \sigma_u^z \sigma_v^z |\psi\rangle = (\cos\theta^{(V)})^{|V|} (\cos\theta^{(U)})^{|U_{odd}|},$$

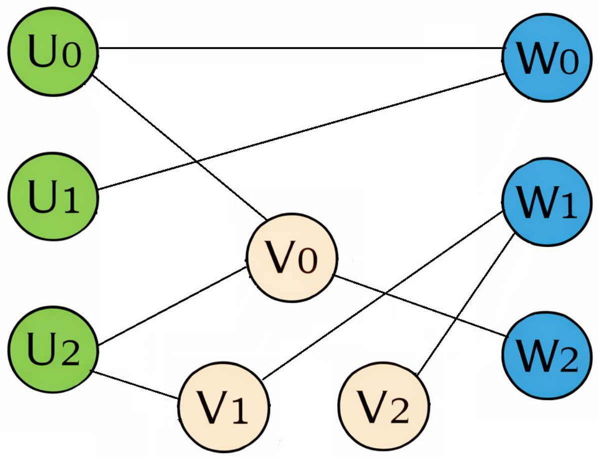
$$\langle\psi| \prod_{v\in V} \sigma_v^z |\psi\rangle = (\cos\theta^{(V)})^{|V|} (\cos\theta^{(U)})^{|U_{even}|}.$$

Here the following notations are used:

$$V_{\text{even}} = \{v \in V \mid \deg_G(v) \equiv 0 \pmod 2\}\,.$$

$$U_{\text{even}} = \{u \in U \mid \deg_G(u) \equiv 0 \pmod 2\}\,, \quad U_{\text{odd}} = \{u \in U \mid \deg_G(u) \equiv 1 \pmod 2\}\,.$$

# Entanglement of quantum states corresponding to tripartite graphs




$$|\psi\rangle = \prod_{(u,v)\in E} \prod_{(v,w)\in E} \prod_{(u,w)\in E} RXY_{uv}(\phi_{uv}) RYZ_{vw}(\phi_{vw}) RXZ_{uw}(\phi_{uw}) |\psi_{init}\rangle ,$$

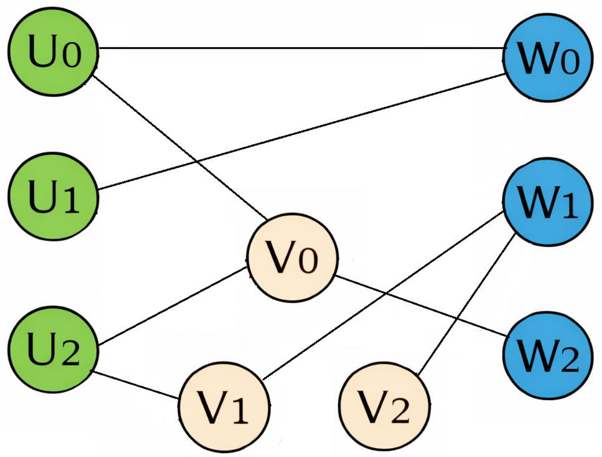
$$|\psi_{init}\rangle = |\psi^{(U)}_{init}\rangle |\psi^{(V)}_{init}\rangle |\psi^{(W)}_{init}\rangle ,$$

$$RXY_{kl}(\phi_{kl}) = e^{-i\frac{\phi_{kl}}{2}\sigma_k^x\sigma_l^y}, \quad RXZ_{kl}(\phi_{kl}) = e^{-i\frac{\phi_{kl}}{2}\sigma_k^x\sigma_l^z}, \quad RYZ_{kl}(\phi_{kl}) = e^{-i\frac{\phi_{kl}}{2}\sigma_k^y\sigma_l^z}.$$

Result for entanglement

$$E_u^{ED}(|\psi\rangle) = \sin^2\theta_u - \left(\sin^2\alpha_u \sin^2\theta_u + \cos^2\theta_u\right) \times$$
$$\prod_{v\in N_V(u)} \prod_{w\in N_W(u)} \left(\cos^2\phi_{uv} + \sin^2\phi_{uv}\sin^2\alpha_v\sin^2\theta_v\right)\left(\cos^2\phi_{uw} + \sin^2\phi_{uw}\sin^2\theta_w\right).$$

# Entanglement of quantum states corresponding to tripartite graphs




$$|\psi\rangle = \prod_{(u,v)\in E} \prod_{(v,w)\in E} \prod_{(u,w)\in E} RXY_{uv}(\phi_{uv}) RYZ_{vw}(\phi_{vw}) RXZ_{uw}(\phi_{uw}) |\psi_{init}\rangle ,$$

$$|\psi_{init}\rangle = |\psi^{(U)}_{init}\rangle |\psi^{(V)}_{init}\rangle |\psi^{(W)}_{init}\rangle ,$$

$$RXY_{kl}(\phi_{kl}) = e^{-i\frac{\phi_{kl}}{2}\sigma^x_k\sigma^y_l}, \quad RXZ_{kl}(\phi_{kl}) = e^{-i\frac{\phi_{kl}}{2}\sigma^x_k\sigma^z_l}, \quad RYZ_{kl}(\phi_{kl}) = e^{-i\frac{\phi_{kl}}{2}\sigma^y_k\sigma^z_l}.$$

Result for the entanglement

$$\begin{aligned} E^{ED}_u(|\psi\rangle) = \sin^2\theta - \left(\sin^2\alpha\sin^2\theta + \cos^2\theta\right) \times \\ \times \left(\cos^2\phi + \sin^2\phi\sin^2\alpha\sin^2\theta\right)^{|N_V(u)|} \times \\ \times \left(\cos^2\phi + \sin^2\phi\sin^2\theta\right)^{|N_W(u)|} . \end{aligned}$$

The entanglement depends on the degrees of vertice u with respect to sets V, W.

## Relation of quantum correlators with structural properties of tripartite graph

$$\langle\psi|\,\sigma^y_{u_1}\sigma^y_{u_2}\,|\psi\rangle =$$

$$= \langle\psi_{init}| \prod_{v_1\in N_V(u_1)} \prod_{v_2\in N_V(u_2)} \prod_{w_1\in N_W(u_1)} \prod_{w_2\in N_W(u_2)} RXY^+_{u_1v_1}(2\phi_{u_1v_1})RXY^+_{u_2v_2}(2\phi_{u_2v_2}) \times$$

$$\times RXZ^+_{u_1w_1}(2\phi_{u_1w_1})RXZ^+_{u_2w_2}(2\phi_{u_2w_2})\sigma^y_{u_1}\sigma^y_{u_2}\,|\psi_{init}\rangle =$$

$$= \frac{1}{2}\Re(-z^{(1)}_{u_1u_2}a_{u_1}a_{u_2} + z^{(2)}_{u_1u_2}a_{u_1}a^*_{u_2}),$$

Quantum graph state

$$|\psi\rangle = \prod_{(u,v)\in E} \prod_{(v,w)\in E} \prod_{(u,w)\in E} RXY_{uv}(\phi_{uv})RYZ_{vw}(\phi_{vw})RXZ_{uw}(\phi_{uw})\,|\psi_{init}\rangle\,,$$

$$|\psi_{init}\rangle = |\psi^{(U)}_{init}\rangle\,|\psi^{(V)}_{init}\rangle\,|\psi^{(W)}_{init}\rangle\,,$$

Notations:

$$a_u = \cos\theta_u - i\sin\alpha_u\sin\theta_u,$$

# Relation of quantum correlators with structural properties of tripartite graph

$$
\begin{aligned}
z^{(1)}_{u_1u_2} = &(\cos\phi + i\sin\theta^{(V)}\sin\alpha^{(V)}\sin\phi)^{|N_V(u_1)\,\triangle\,N_V(u_2)|}\times\\
&\times(\cos\phi + i\cos\theta^{(W)}\sin\phi)^{|N_W(u_1)\,\triangle\,N_W(u_2)|}\times\\
&\times(\cos(2\phi) + i\sin\theta^{(V)}\sin\alpha^{(V)}\sin(2\phi))^{|N_V(u_1)\cap N_V(u_2)|}\\
&(\cos(2\phi) + i\cos\theta^{(W)}\sin(2\phi))^{|N_W(u_1)\cap N_W(u_2)|}
\end{aligned}
$$

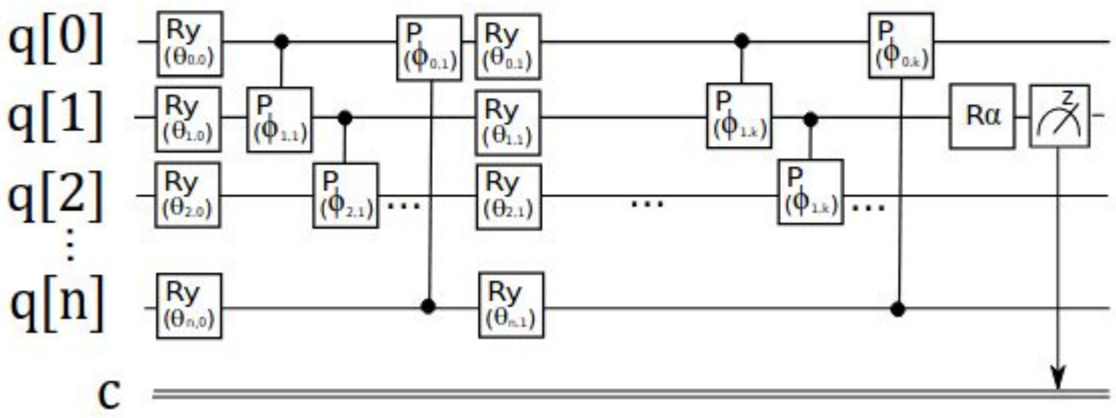
$$
\begin{aligned}
z^{(2)}_{u_1u_2} = &(\cos\phi + i\sin\theta^{(V)}\sin\alpha^{(V)}\sin\phi)^{|N_V(u_1)/N_V(u_2)|}\times\\
&\times(\cos\phi - i\sin\theta^{(V)}\sin\alpha^{(V)}\sin\phi)^{|N_V(u_2)/N_V(u_1)|}\times\\
&\times(\cos\phi + i\cos\theta^{(W)}\sin\phi)^{|N_W(u_1)/N_W(u_2)|}(\cos\phi - i\cos\theta^{(W)}\sin\phi)^{|N_W(u_2)/N_W(u_1)|}.
\end{aligned}
$$

Here $|N_A(u_1)\,\triangle\,N_A(u_2)|$ is the cardinality of the symmetric difference of the sets $N_A(u_1)$, $N_A(u_2)$ which corresponds to the number of different (non-overlapping) neighbors of $u_1$, $u_2$; and $|N_A(u_1)\cap N_A(u_2)|$ is equal to the number of common neighbors of $u_1$, $u_2$ in $A$, ($A$ denotes $V, W$ in tripartite graph $G(U,V,W,E)$). It is important to note that the value $|N_A(u_1)\cap N_A(u_2)|$ determines the number of 4-cycles involving the vertices $u_1$, $u_2$, and vertices from the set $A$ in the tripartite graph. Namely the number reads $n_4^{(A)} = C^2_{|N_A(u_1)\cap N_A(u_2)|}$.

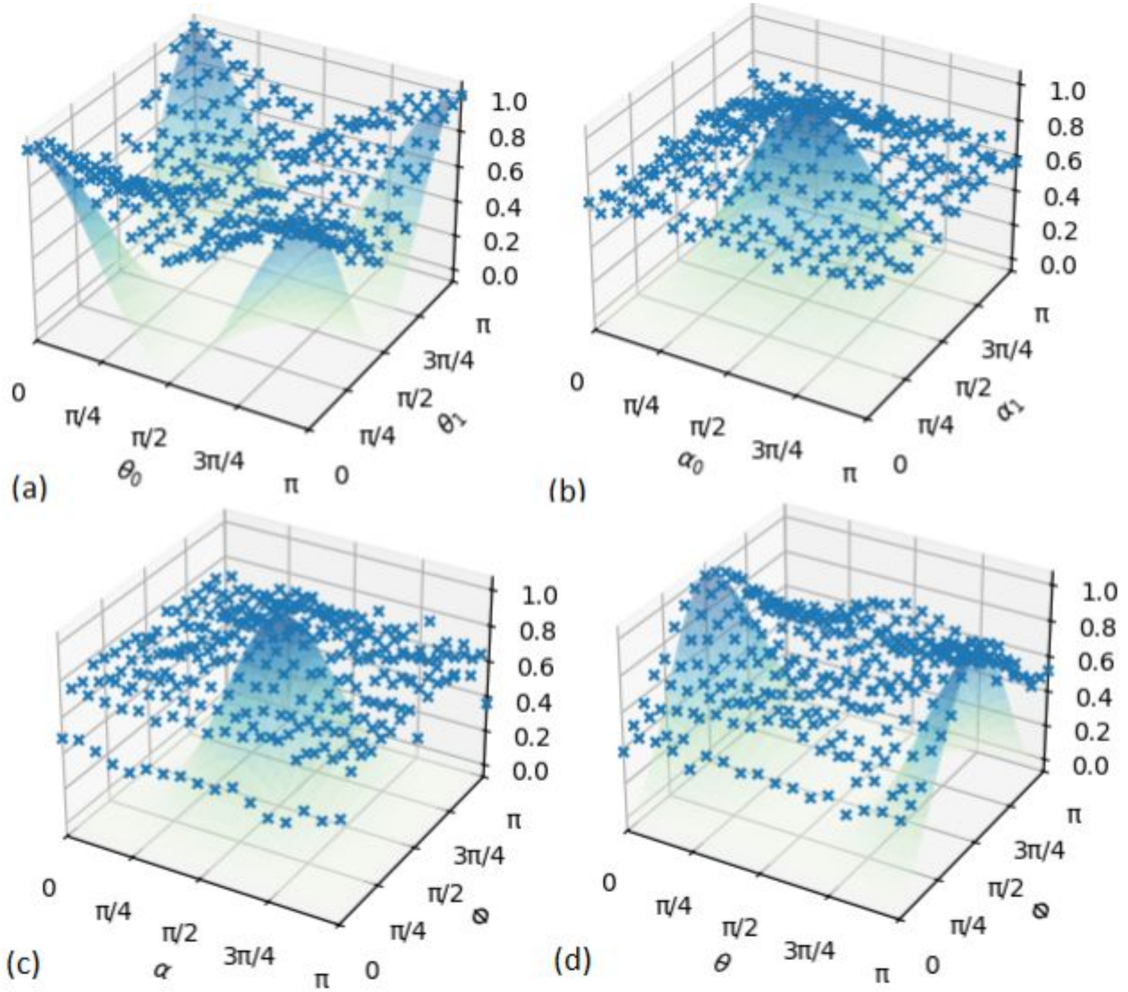
## Importance of research on quantum states representing complex structures

- preparation of entangled quantum states;
- quantum error correction algorithms;
- study of properties of complex systems with quantum programming;
- quantum machine learning.

# Quantum machine learning. Variational quantum states




Entanglement of VQS




Kh. Gnatenko

IEEE International Conference on Quantum Computing and Engineering (QCE), Albuquerque, NM, USA, 2025, 470–471 (2025)

Thank you for your attention!