Spectral properties of random Markov generators: classical vs quantum

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Spectral properties of random Markov generators: classical vs quantum

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In collaboration with

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Outline

1. Basic introduction to classical and quantum Markov generators
2. Basic spectral properties of maps and generators
3. Random channels: classical and quantum
4. Random generators: classical and quantum
5. Random Matrix Models
6. Dissipative quantum chaos
7. Summary

S. Denysov, T. Lapteva, W. Tarnowski, DC, K. Życzkowski, Phys.Rev.Lett. (2019)
W. Tarnowski, I. Yusipov, T. Lapteva, S. Denysov, DC, K. Życzkowski, PRE (2021)
W. Tarnowski, DC, S. Denysov, K. Życzkowski, OSID (2023)
DC, S. Denysov. W. Tarnowski, K. Życzkowski, J. Phys. A (2026)

- classical

$$\dot{\boldsymbol{p}} = K\boldsymbol{p}$$

$$\boldsymbol{p} \longrightarrow S(t)\boldsymbol{p}\ ; \quad S(t) = e^{tK} = \text{classical channel}$$

$$S(t_1)S(t_2) = S(t_1+t_2)\ ; \ \text{classical channel} = \text{stochastic matrix}$$

- quantum

$$\dot{\boldsymbol{\rho}} = \mathcal{L}\boldsymbol{\rho}$$

$$\boldsymbol{\rho} \longrightarrow \Lambda(t)\boldsymbol{\rho}\ ; \quad \Lambda(t) = e^{t\mathcal{L}} = \text{quantum channel}$$

$$\Lambda(t_1)\Lambda(t_2) = \Lambda(t_1+t_2)\ ; \quad \text{quantum channel} = \text{CPTP map}$$

Markov generators

- classical (Kolmogorov conditions)

$$K_{ij} \geq 0 \ (i \neq j) \ , \quad \sum_i K_{ij} = 0$$

$$K_{ij} = R_{ij} - \delta_{ij} \, R_j \ ; \quad R_j = \sum_k R_{kj} \ ; \quad R_{ij} \geq 0$$

- quantum: Gorini-Kossakowski-Lindblad-Sudarshan (1976)

$$\mathcal{L}(\boldsymbol{\rho}) \ = \ -i[H, \boldsymbol{\rho}] + \mathcal{L}_D(\boldsymbol{\rho})$$

$$\mathcal{L}_D(\boldsymbol{\rho}) = \sum_k \gamma_k \left(V_k \boldsymbol{\rho} V_k^\dagger - \frac{1}{2} \{ V_k^\dagger V_k, \boldsymbol{\rho} \} \right) \ ; \quad \gamma_k > 0$$

Why CP?

- positive: $X \geq 0 \;\Rightarrow\; \Phi(X) \geq 0$
- k-positive: $\mathcal{I}_k \otimes \Phi$ is positive $(\mathbb{C}^k \otimes \mathcal{H})$
- completely positive: k-positive for $k = 1, 2, \ldots$

$$\Phi_1 \otimes \Phi_2 \quad ???$$

Positive $\otimes$ Positive need not be positive

CP $\otimes$ CP always CP

Classical vs quantum

Fix $\{|1\rangle, \ldots, |d\rangle\}$ (orthonormal basis)

$$K_{ij} := \langle i| \mathcal{L}(|j\rangle\langle j|) |i\rangle$$

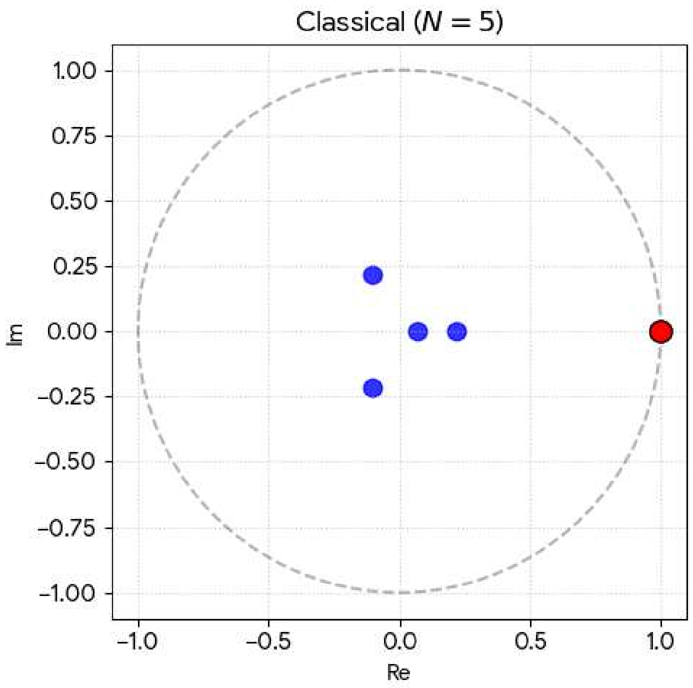
defines a classical generator. Similarly, if Φ is a quantum channel

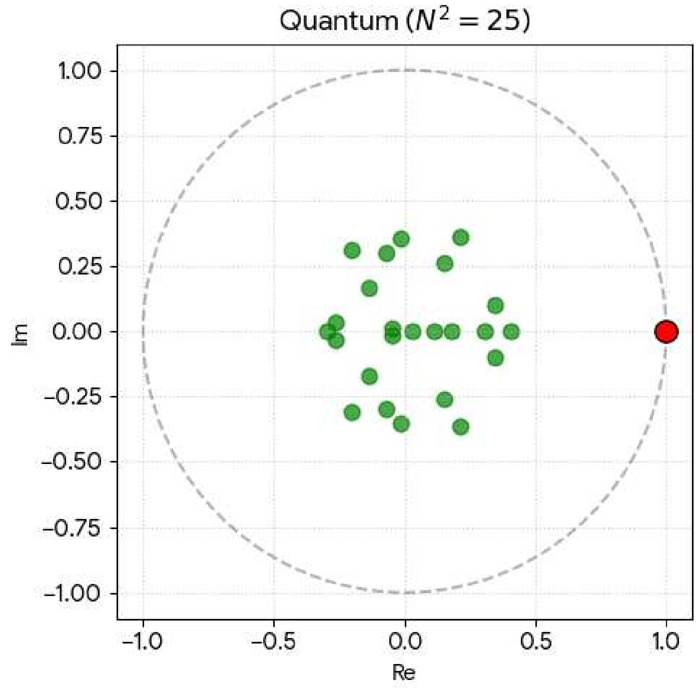
$$S_{ij} = \langle i| \Phi(|j\rangle\langle j|) |i\rangle$$

defines a stochastic matrix.

Perron-Frobenius: classical and quantum channels

$$|\lambda| \leq 1$$

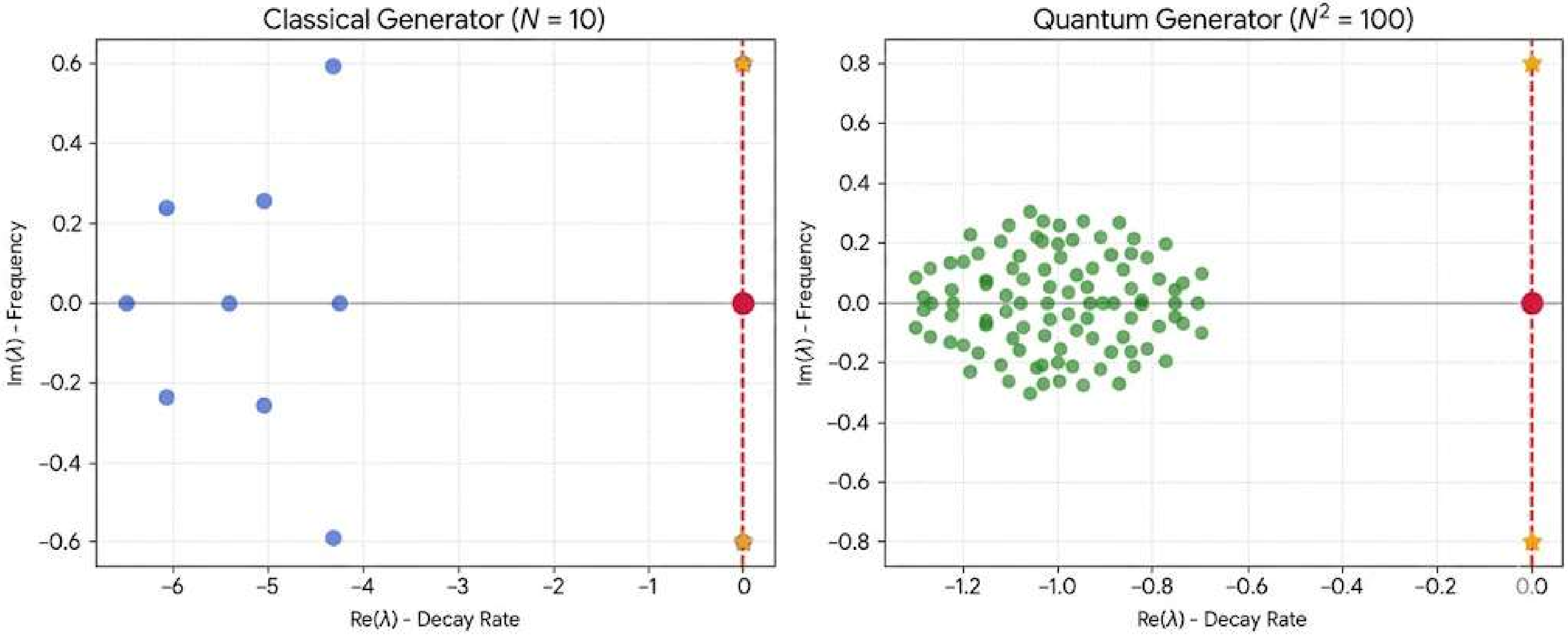




(still holds for positive trace-preserving maps)

Spectra of generators

$$\operatorname{Re}\lambda \leq 0$$



Gershgorin vs. Brauer

Let T_{jk} be a stochastic matrix such that a_1 is the smallest and a_2 the second smallest diagonal entries.

- all eigenvalues belong to the Perron-Frobenius disc $|z| \leq 1$
- all eigenvalues belong to the largest Gershgorin disc

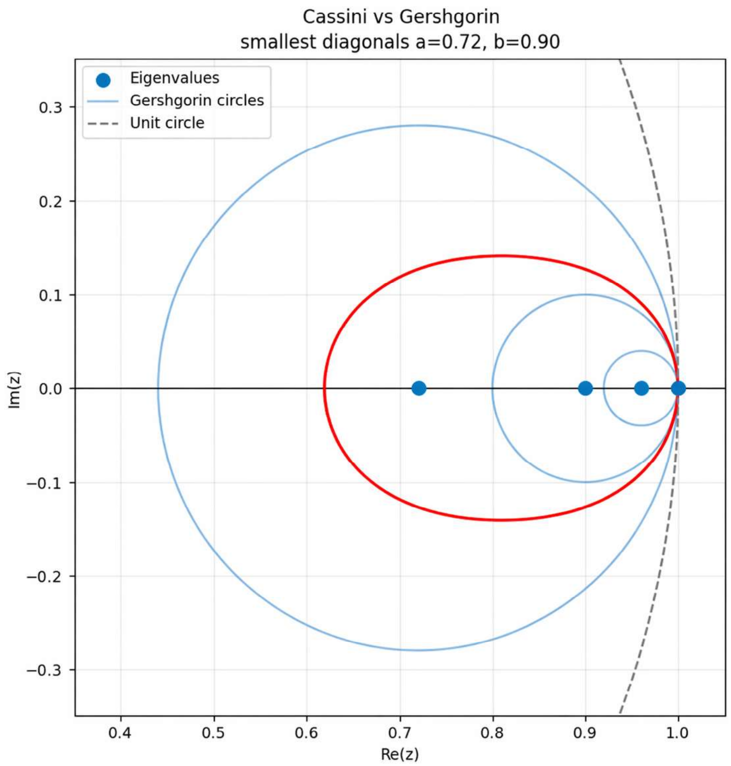
$$|z - a_1| \leq 1 - a_1$$

- all eigenvalues of T belong to the Cassini oval

$$|z - a_1| \cdot |z - a_2| \leq (1 - a_1) \cdot (1 - a_2)$$

S. Gerschgorin, Isv. Akad. Nauk USSR Ser. Mat. **7**, 749 (1931)
A. Brauer, Duke Math. J. **14**, 21 (1947)

$$T = \begin{pmatrix} a & 0 & 0 & 0 \\ 1-a & b & 0 & 0 \\ 0 & 1-b & c & 0 \\ 0 & 0 & 1-c & 1 \end{pmatrix} \quad ; \quad \sigma(T) = \{1, a, b, c\}$$



Detailed balance

- Classical: $K\boldsymbol{\pi} = 0$

$$K_{ij}\pi_j = K_{ji}\pi_i \quad ; \qquad \textbf{(real spectrum!)}$$

- quantum: $\mathcal{L}(\boldsymbol{\omega}) = 0$
  - $[H, \boldsymbol{\omega}] = 0$
  - $(A, B)_{\boldsymbol{\omega}} = \mathrm{Tr}(\boldsymbol{\omega} A^\dagger B)$

$$\mathcal{L}_D^\dagger = \text{Heisenberg picture}$$

$$(\mathcal{L}_D^\dagger(A), B)_{\boldsymbol{\omega}} = (A, \mathcal{L}_D^\dagger(B))_{\boldsymbol{\omega}}$$

$\mathcal{L}_D^\dagger$ is Hermitian and hence it has a **real spectrum!**

Classical generator

Let $\mu := \max_i |K_{ii}|$

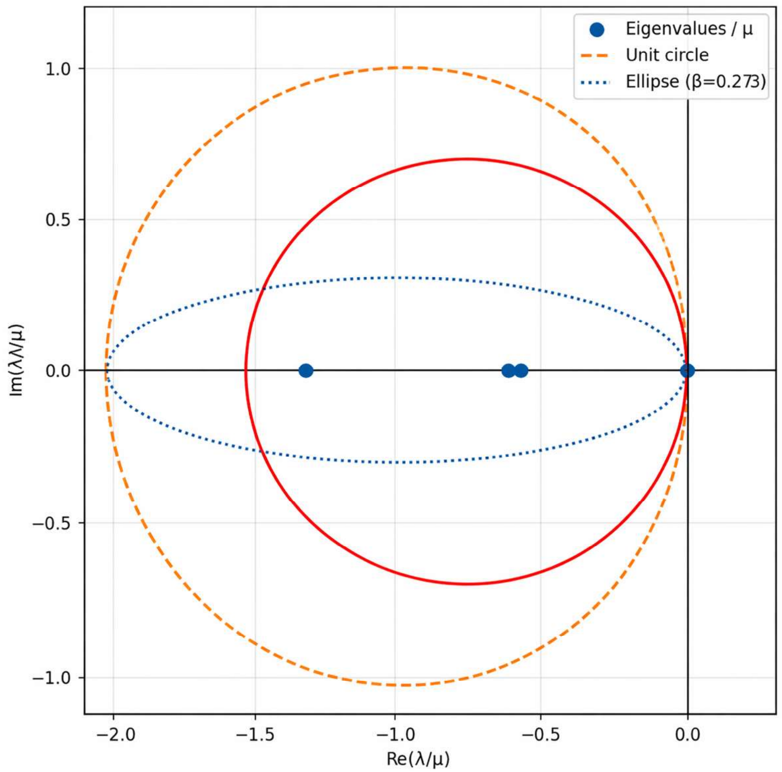
- eigenvalues of $\frac{1}{\mu}K$ belong to a disc $\mathbb{D}(-1, 1)$
- $\frac{1}{\mu}K + I$ = stochastic matrix $\rightarrow$ Cassini oval
- thermodynamical constraint: Let $K\boldsymbol{\pi} = 0$

$$F_{ij} = \ln \frac{K_{ij}\pi_j}{K_{ji}\pi_i} \ , \qquad \beta := \mathrm{tgh}(\max_{ij} F_{ij}/2)$$

all eigenvalue of $\frac{1}{\mu}K$ belong to the ellipse

$$(x+1)^2 + (y/\beta)^2 \leq 1$$

Xu et al, Phys. Rev. Lett. **135**, 257102 (2025)



$\sigma(K) \subset$ thermodynamic ellipse $\cap$ Cassini oval

Quantum generator

$$K_{ii} \leq 0 \; ; \quad \mu = \max_i |K_{ii}|$$

All eigenvalues of $\frac{1}{\mu}K$ belong to the unit disc $\mathbb{D}(-1,1)$

$$\kappa := \max_\psi \langle\psi|\, \mathcal{L}_D(|\psi\rangle\langle\psi|)\, |\psi\rangle \; ; \quad \|\psi\| = 1$$

All eigenvalues of $\frac{1}{\kappa}\mathcal{L}_D$ belong to the unit disc $\mathbb{D}(-1,1)$

DC, S. Denisov. W. Tarnowski, K. Życzkowski, J. Phys. A (2026)

Bound on relaxation rates

$$\mathrm{Re}\,\lambda_k \leq 0 \quad \rightarrow \quad \Gamma_k := -\mathrm{Re}\,\lambda_k$$

$$\Gamma := \Gamma_1 + \ldots + \Gamma_{N^2-1}$$

$$\Gamma_k \leq \frac{1}{N}\,\Gamma$$

- universal
- tight
- purely quantum
- only 2-positivity is essential!

DC, F. vom Ende, G. Kimura. P. Muratore-Ginanneschi (2025)

What are spectral properties of typical (= random) classical and quantum dissipative generators?

Random quantum channel

$$\Phi : \mathcal{M}_N \to \mathcal{M}_N$$

- completely positive
- trace-preserving

$$\Phi(X) = \sum_\alpha K_\alpha X K_\alpha^\dagger$$

$$\sum_\alpha K_\alpha^\dagger K_\alpha = \mathbb{1}$$

$$\Phi \;\to\; \mathbf{C} = \sum_{i,j=1}^{N} |i\rangle\langle j| \otimes \Phi(|i\rangle\langle j|) \geq 0$$

$$\Phi \quad \longleftrightarrow \quad \mathbf{C} = \sum_{i,j} |i\rangle\langle j| \otimes \Phi(|i\rangle\langle j|)$$

$$\mathbf{C} \geq 0 \quad ; \quad \mathrm{Tr}_2\mathbf{C} = \mathbb{1}$$

$$\mathbf{G} \in \mathcal{M}_{N^2} \quad ; \quad \mathbf{G}_1 = \mathrm{Tr}_2\mathbf{G}\mathbf{G}^\dagger$$

$$\mathbf{C} = (\mathbf{G}_1^{-1/2} \otimes \mathbb{1})\,\mathbf{G}\mathbf{G}^\dagger\,(\mathbf{G}_1^{-1/2} \otimes \mathbb{1})$$

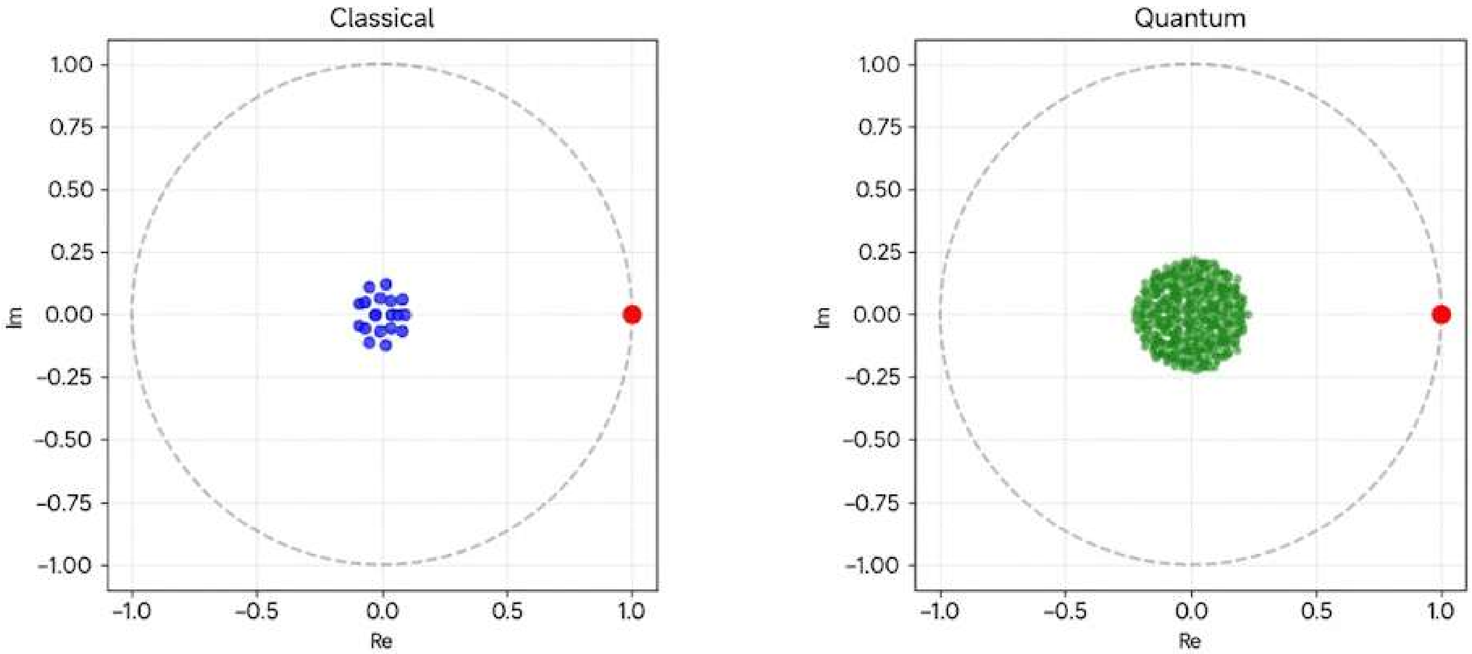
random Φ $\longleftrightarrow$ random $\mathbf{G}$

$\mathbf{G} \in N^2 \times N^2$ complex Ginibre ensemble

G_{ij} i.i.d. $\mathcal{N}(0,1) \;\rightarrow\; \mathbf{GG}^\dagger$ – Wishart matrix

$N \rightarrow \infty$ circular Girko law

- spectral density Φ is uniform on the $r = \frac{1}{N}$ disk
- spectral density T is uniform on the $r = \frac{1}{\sqrt{N}}$ disk



How to explain the spectrum by Random Matrix Theory?

Random matrices ...

- Wigner's approach to nuclear physics
- Dyson's statistical theory of spectra
- classical and quantum chaos
- 2D quantum gravity
- large N limit of gauge theory
- communication systems
- finances
- ...

RMT

Hermitian case — Dyson threefold way

- $\beta = 1$ — Gaussian Orthogonal Ensemble (**GOE**)
- $\beta = 2$ — Gaussian Unitary Ensemble (**GUE**)
- $\beta = 4$ — Gaussian Symplectic Ensemble (**GSE**)

Non-Hermitianian matrices

$$\Phi^\dagger \neq \Phi$$

Bloch vector representation of any state ρ

$$\rho = \frac{1}{N}\mathbb{1} + \sum_{\alpha=1}^{N^2-1} x_\alpha \, \sigma_\alpha$$

where σ_α are generators of SU(N) such that $\mathrm{tr}\,(\sigma_\alpha \sigma_\beta) = \delta_{\alpha\beta}$

Generalized Bloch vector $\mathbf{x} = [x_1, \ldots, x_{N^2-1}] \in \mathbb{R}^{N^2-1}$

Quantum channel Φ in the Bloch representation

$$\rho' = \Phi(\rho) \quad \rightarrow \quad \mathbf{x}' = \mathbf{T}\mathbf{x} + \mathbf{r},$$

that is,

Φ is represented by $N^2 \times N^2$ real matrix $\left(\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{r} & \mathbf{T} \end{array} \right)$

$$\mathrm{spec}(\Phi) \; = \; \mathrm{spec}(\mathbf{T}) \cup \{1\}$$

Random Matrix Model for $\mathbb{T}$

$T \in$ real Ginibre ensemble G_{N^2-1} ;

Girko circular law $\sim \dfrac{1}{N}$

$$\rho' = \Phi(\rho) \quad \rightarrow \quad \mathbf{x}' = \mathbf{T}\mathbf{x} + \mathbf{r},$$

Complete positivity is not essential!

Lindbladian

$$\mathcal{L}(\rho) \;=\; -i[H,\rho] + \mathcal{L}_D(\rho)$$

$$\mathcal{L}_D(\rho) = \sum_{k,\ell=1}^{N^2-1} K_{k\ell} \left(F_k \rho F_\ell^\dagger - \frac{1}{2}\{F_\ell^\dagger F_k, \rho\} \right)$$

Kossakowski matrix: $[K_{k\ell}] \geq 0$, $\mathrm{Tr}(F_k F_\ell^\dagger) = \delta_{k\ell}$

Diagonal form

$$\mathcal{L}_D(\rho) = \sum_\alpha \gamma_\alpha \left(V_\alpha \rho V_\alpha^\dagger - \frac{1}{2}\{V_\alpha^\dagger V_\alpha, \rho\} \right)$$

Random generators

$$\mathcal{L}(\rho) = -i[H,\rho] + \sum_{k,l=1}^{N^2-1} K_{kl}\left(F_k \rho F_l^\dagger - \frac{1}{2}\{F_l^\dagger F_k, \rho\}\right)$$

Random H ; Random K_{ij}

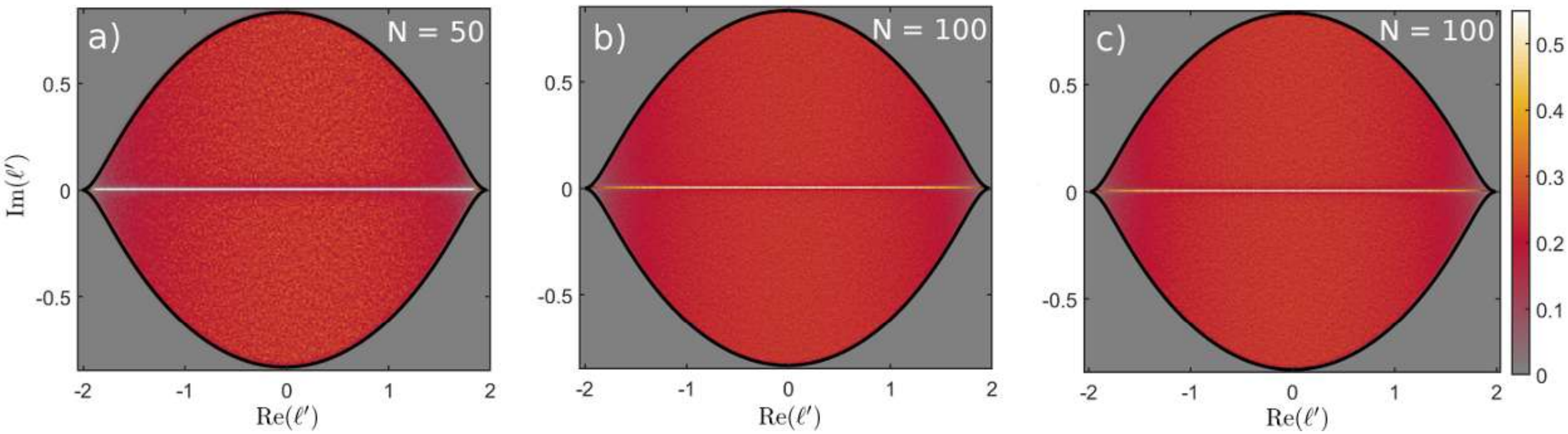
$H \in$ **GUE** ; $K \in$ **random Wishart**

$K = G^\dagger G$; $G \in$ **complex Ginibre**

Strong dissipation ($H = 0$)

$$\mathcal{L}(\rho) = \sum_{k,l=1}^{N^2-1} K_{kl} \left(F_k \rho F_l^\dagger - \frac{1}{2}\{F_l^\dagger F_k, \rho\} \right)$$

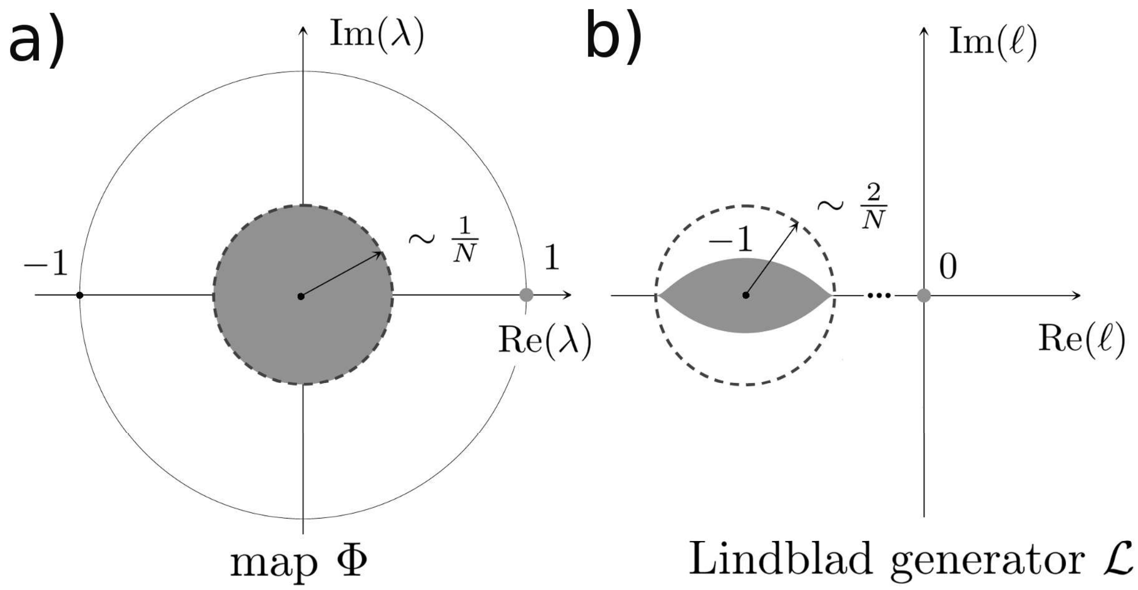
$$\langle \mathrm{Tr} K \rangle = N \quad ; \quad N(\mathcal{L} + \mathbb{1})$$



S. Denisov, T. Lapteva, W. Tarnowski, DC, K. Życzkowski, Phys.Rev.Lett. (2019)

Spectra of a random map Φ and generator $\mathcal{L}$

a) spectrum of a **random map** Φ consist of the F-P leading eigenvalue, $\lambda_0 = 1$ and a **disk of eigenvalues** of radius $R \sim 1/N$



b) spectrum of a random **Lindblad generator** consists in $\lambda_0 = 0$ and a **lemon of eigenvalues**
spectral gap $\sim 1 - \frac{2}{N}$

S. Denysov, T. Lapteva, W. Tarnowski, DC, K. Życzkowski, Phys.Rev.Lett. (2019)

Random Matrix Model

$$\mathcal{L}(\rho) = \sum_{k=1}^{N^2-1} \left(V_k \rho V_k^\dagger - \frac{1}{2}\{V_k^\dagger V_k, \rho\} \right)$$

vectorization: $\mathcal{L} \rightarrow \widehat{\mathcal{L}}$

$$\widehat{\mathcal{L}} = \widehat{\Phi} - \frac{1}{2}(B \otimes \mathbb{1} + \mathbb{1} \otimes \overline{B})$$

$$\widehat{\Phi} = \sum_k V_k \otimes \overline{V_k} \; ; \quad B = \sum_k V_k^\dagger V_k$$

Random Matrix Model

$$\widehat{\mathcal{L}} = \widehat{\Phi} - \frac{1}{2}(B \otimes \mathbb{1} + \mathbb{1} \otimes \overline{B})$$

If Φ is trace preserving, then $B = \mathbb{1}$ and hence $\widehat{\mathcal{L}} = \widehat{\Phi} - \mathbb{1} \otimes \mathbb{1}$. In the general case the dual map $\Phi^\dagger$ is **not unital**, so

$$B := \Phi^\dagger(\mathbb{1}) = \mathbb{1} + X$$

where a Hermitian matrix $X = X^\dagger \neq 0$.

$$\widehat{\mathcal{L}} = \widehat{\Phi} - \mathbb{1} \otimes \mathbb{1} - \frac{1}{2}(X \otimes \mathbb{1} + \mathbb{1} \otimes \overline{X})$$

Random Matrix Model

$$\widehat{\mathcal{L}} = \widehat{\Phi} - \mathbb{1} \otimes \mathbb{1} - \frac{1}{2}(X \otimes \mathbb{1} + \mathbb{1} \otimes \overline{X})$$

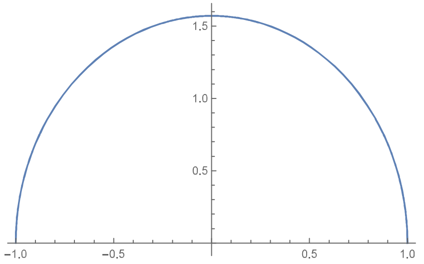
$$\widehat{\mathcal{L}} \;\to\; \widehat{\mathcal{L}}' = N(\widehat{\mathcal{L}} + \mathbb{1} \otimes \mathbb{1})$$

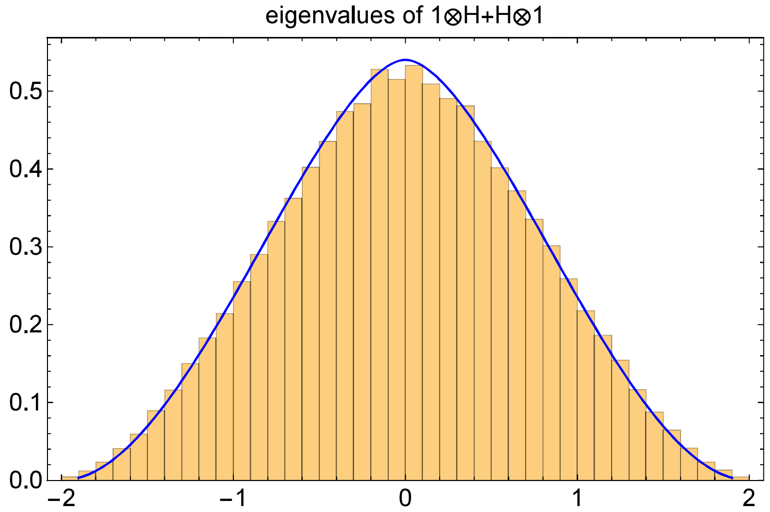
$$\widehat{\mathcal{L}}_{\mathrm{RMT}} = G_R - \mathbb{1} \otimes \mathbb{1} - \frac{1}{2}(C \otimes \mathbb{1} + \mathbb{1} \otimes C)$$

$G_R \in$ **real Ginibre** ; $\;C \in \mathbf{GOE}$

$C \otimes \mathbb{1} + \mathbb{1} \otimes C$ (convolution)

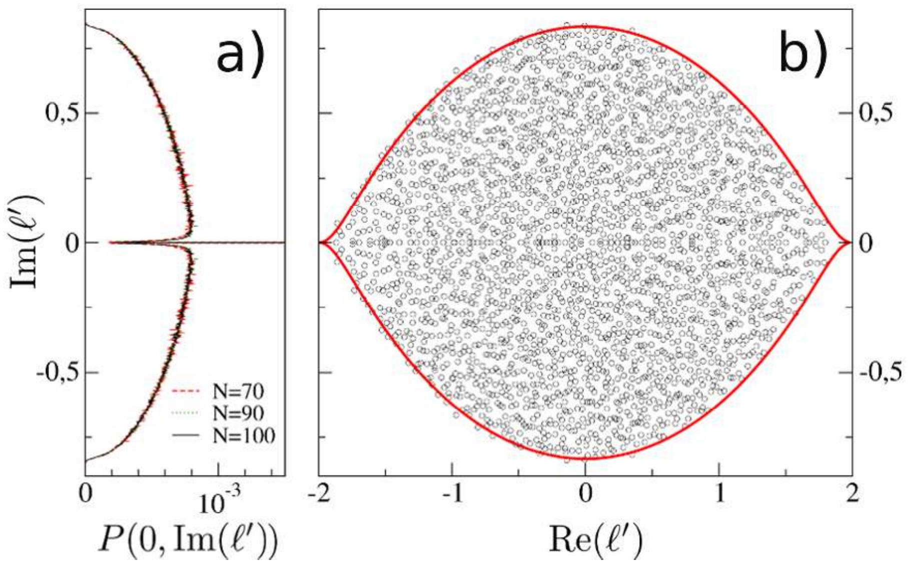
GOE $\rightarrow \quad \rho(x) = \dfrac{1}{2\pi}\sqrt{1-x^2}\ ; \quad |x| \leq 1$ (**Wigner semicircle**)





$G_R - (C \otimes \mathbb{1} + \mathbb{1} \otimes C)$

G_R from **real Ginibre** and $(C \otimes \mathbb{1} + \mathbb{1} \otimes C)$ with C from **GOE** are **asymptotically free**.
Free convolution of Girco disc with classical convolution of two Wigner semicircles gives



density $=?$; boundary $=?$

Boundary from RMM

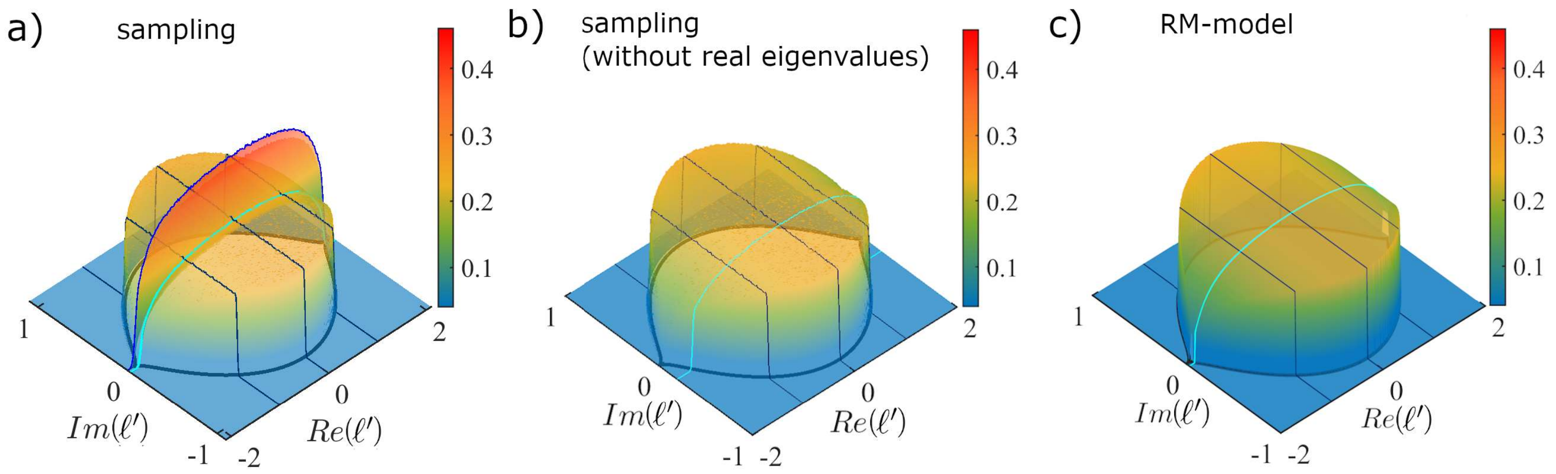
The boundary of the spectrum of $\mathcal{L}'_{\mathrm{RMT}}$ is given by the solution of equation for complex z

$$\mathsf{Im}[z + G(z)] = 0, \quad \text{with}$$

$$G(z) = 2z - \frac{2z}{3\pi}\left[(4+z^2)E\left(\frac{4}{z^2}\right) + (4-z^2)K\left(\frac{4}{z^2}\right)\right],$$

$E(k)$ and $K(k)$ are complete elliptic integrals of the first and second kind.

Density from RMM



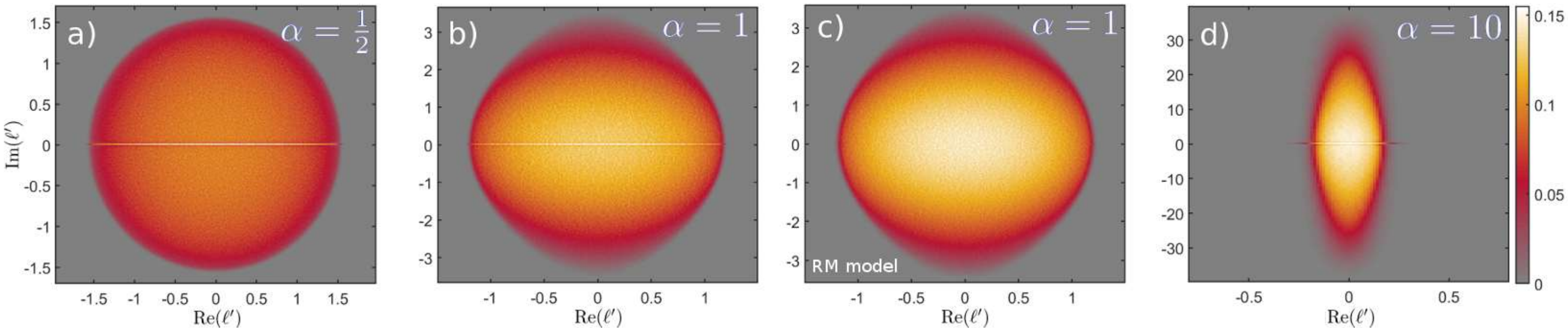
$$z = x + iy \; ; \quad \rho(x, y) = \rho(x)$$

W. Tarnowski, I. Yusipov, T. Lapteva, S. Denisov, DC, K. Życzkowski, PRE 2021

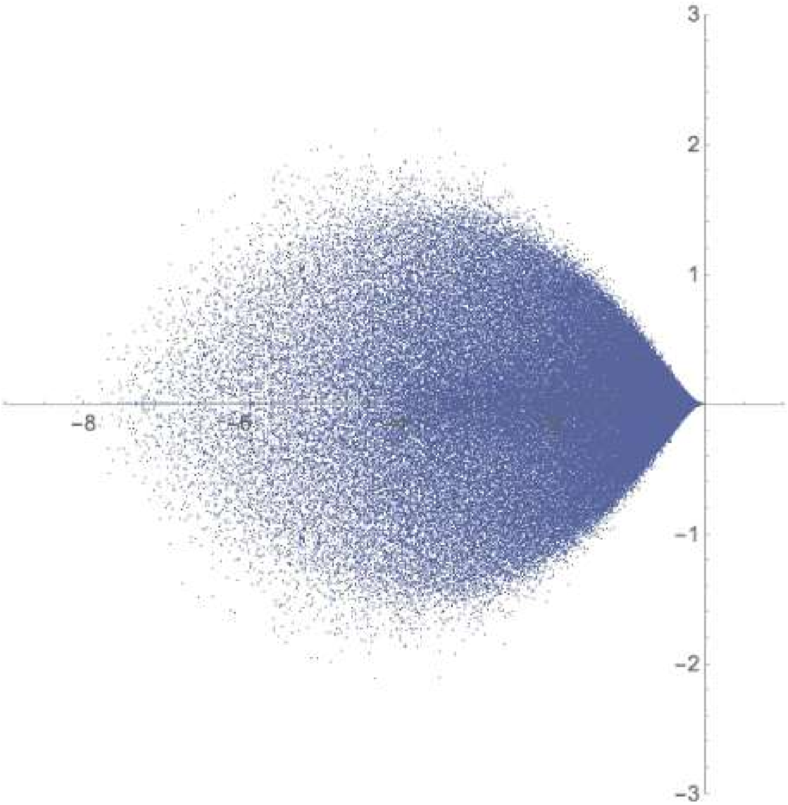
Adding a Hamiltonian part

$$G_R - \frac{1}{2}(W \otimes \mathbb{1} + \mathbb{1} \otimes \overline{W})$$

$$W = C + i\alpha H \ ; \quad C \in \textbf{GOE} \ ; \ H \in \textbf{GUE}$$



Single jump operator



No spectral gap!

T. Can, J. Phys. A (2019)

Many body interaction

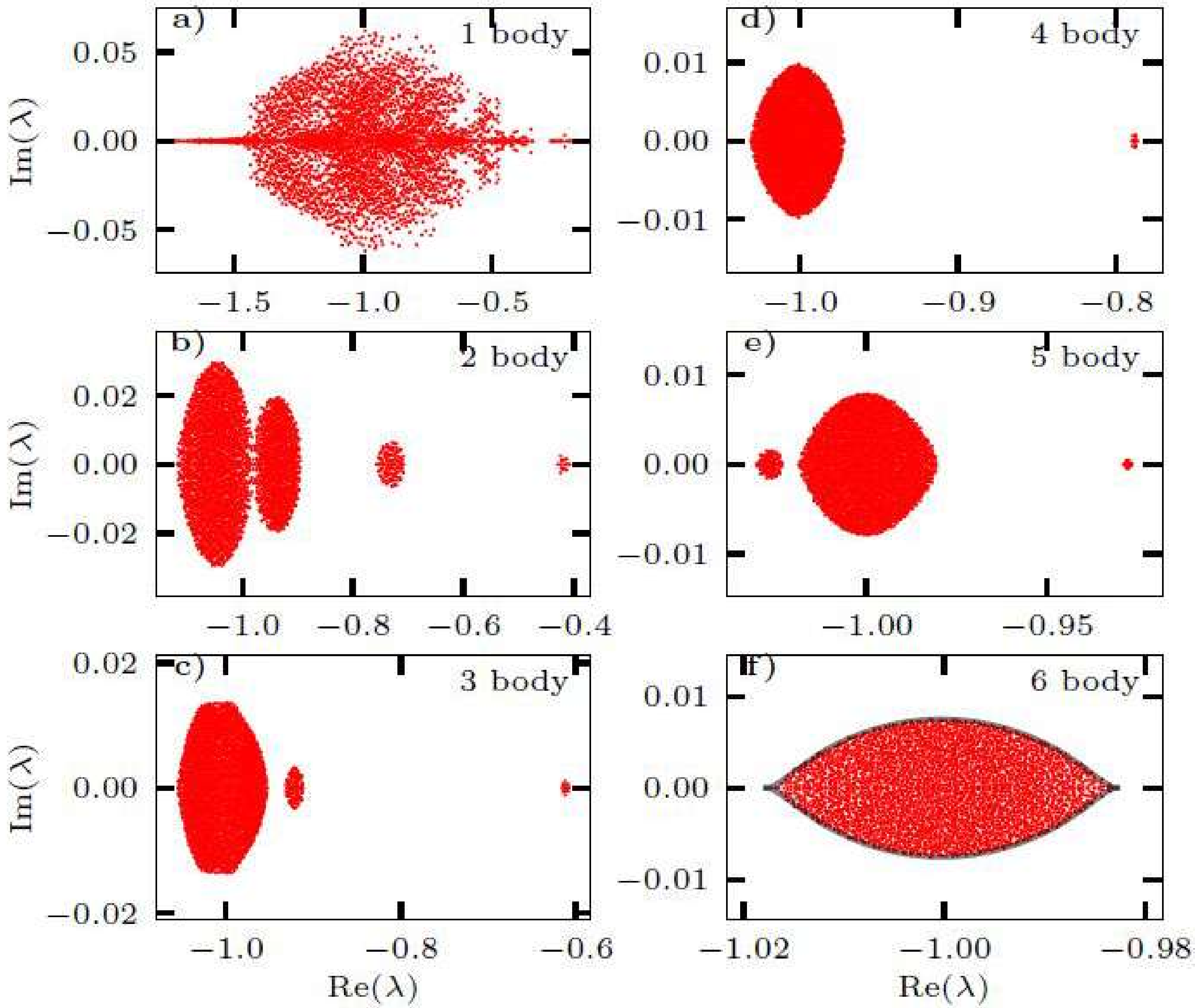
$$\mathcal{L}_D(\rho) = \sum_{i,j} K_{ij} \left(L_i \rho L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_i, \rho\} \right)$$

ℓ qubits

$$L_i \propto \sigma_{x_1} \otimes \sigma_{x_2} \otimes \ldots \otimes \sigma_{x_\ell}$$

$$x_i \in \{0,1,2,3\} \quad ; \quad (\sigma_0 = \mathbb{1})$$

k–body interaction $=$ at most k nonzero x_i



K. Wang, F. Piazza, and D. J. Luitz, PRL 2019

Classical Kolmogorov generator

$$K_{ij} = R_{ij} - \delta_{ij}\, R_j \ ; \qquad R_j = \sum_k R_{kj}$$

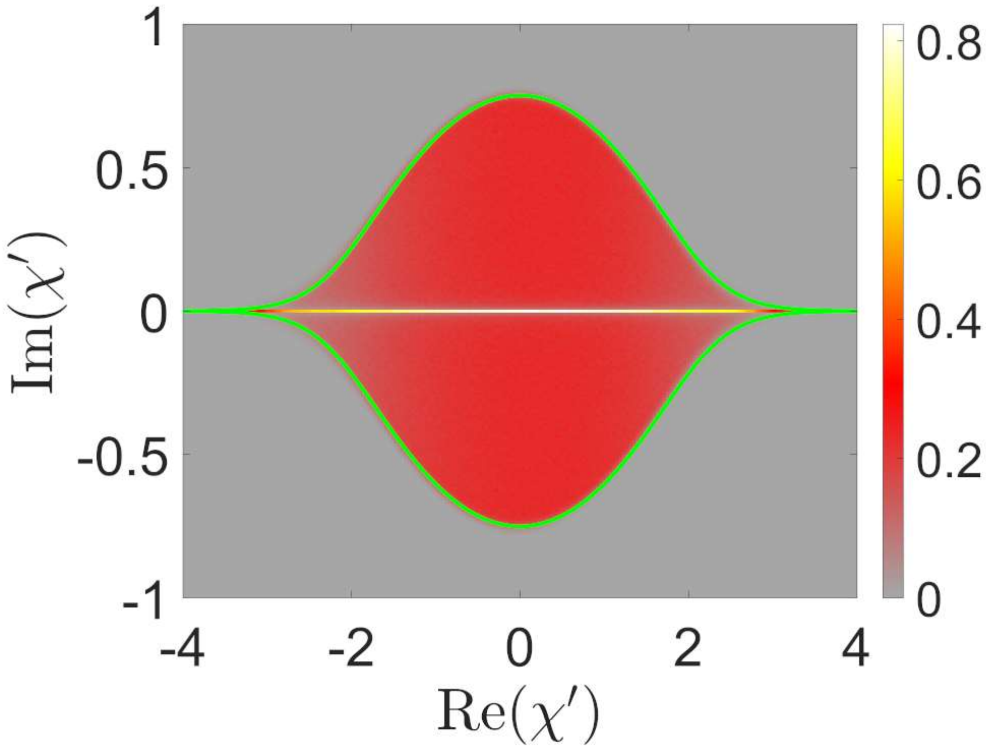
$$R_{ij} \geq 0$$

$$R_{ij} = |z_{ij}|^2$$

z_{ij} are i.i.d. Gaussian complex variables

Classical Kolmogorov generator

RMM: $G_{\mathrm{R}} + D \quad \to \quad N^{3/2}(\lambda + 1)$



C. Timm, PRE 2009; S. Lange and C. Timm, Chaos 2021
W. Tarnowski, I. Yusipov, T. Lapteva, S. Denisov, DC, K. Życzkowski, PRE 2021

Integrable vs. chaotic

- **Classical chaos**: tiny differences in initial conditions grow exponentially (positive Lyapunov exponent),
- **Quantum chaos**: classical limit is chaotic; revealed not by trajectory sensitivity but by signatures like random-matrix spectral statistics or eigen-state delocalization.
- **Dissipative quantum chaos** (chaotic behavior in open quantum systems): quantum-map/quantum Liouvillian spectral statistics

Quantum chaos – Hermitian case

Nearest-neighbor spacing $P(s)$

- Poisson (integrable, many-body localized)

$$P(s) \sim e^{-s} \qquad \text{(no level repulsion)}$$

- Wigner-Dyson (quantum chaotic):

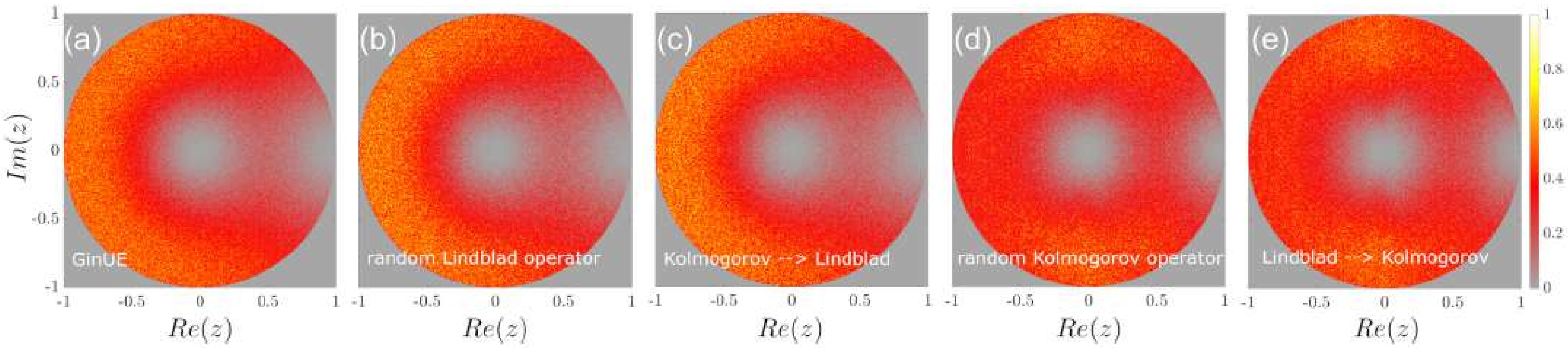
$$P(s) \sim s^{\beta} e^{-cs^2} \qquad \text{(level repulsion)}$$

Dissipative quantum chaos – non-Hermitian case

Complex Spacing Ratio

$$z_k := \frac{\lambda_k^{\mathrm{NN}} - \lambda_k}{\lambda_k^{\mathrm{NNN}} - \lambda_k}$$

Sá, Ribeiro, Prosen, PRX 2020



W. Tarnowski, I. Yusipov, T. Lapteva, S. Denisov, DC, K. Życzkowski, PRE 2021

Detailed balance

- Classical: $K\boldsymbol{p} = 0$

$$K_{ij}p_j = K_{ji}p_i$$

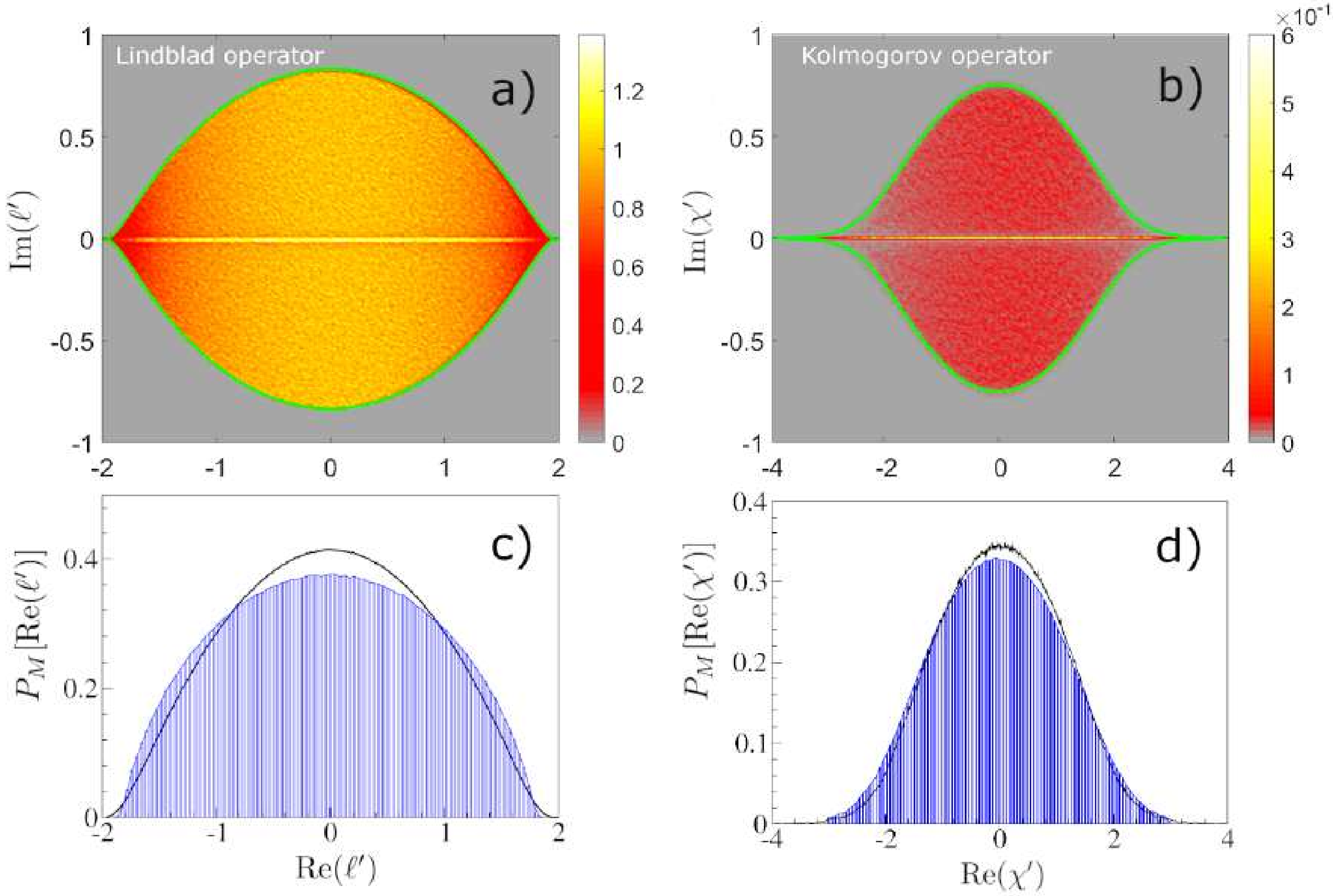
- quantum: $\mathcal{L}(\boldsymbol{\omega}) = 0$
  - $[H, \boldsymbol{\rho}] = 0$
  - $(A, B)_{\boldsymbol{\omega}} = \mathrm{Tr}(\boldsymbol{\omega} A^\dagger B)$

$$\mathcal{L}_D^\dagger = \text{Heisenberg picture}$$

$$(\mathcal{L}_D^\dagger(A), B)_{\boldsymbol{\omega}} = (A, \mathcal{L}_D^\dagger(B))_{\boldsymbol{\omega}}$$

$\mathcal{L}_D^\dagger$ is Hermitian and hence it has a real spectrum!

Detailed balance



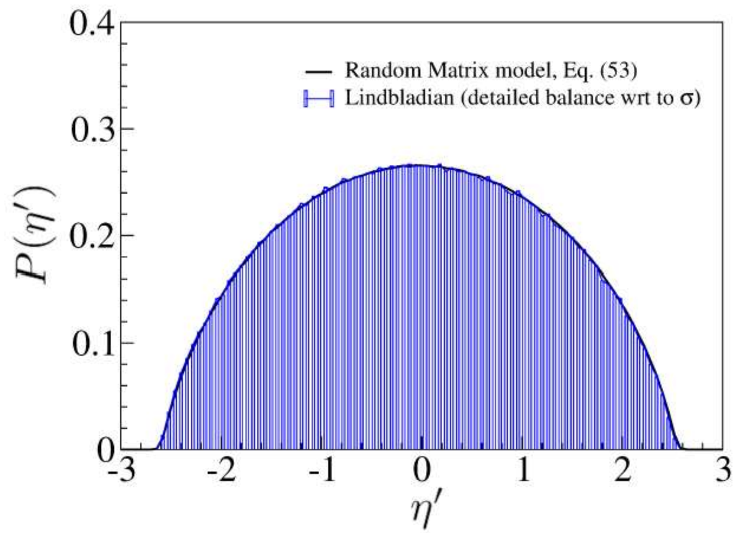
Detailed balance

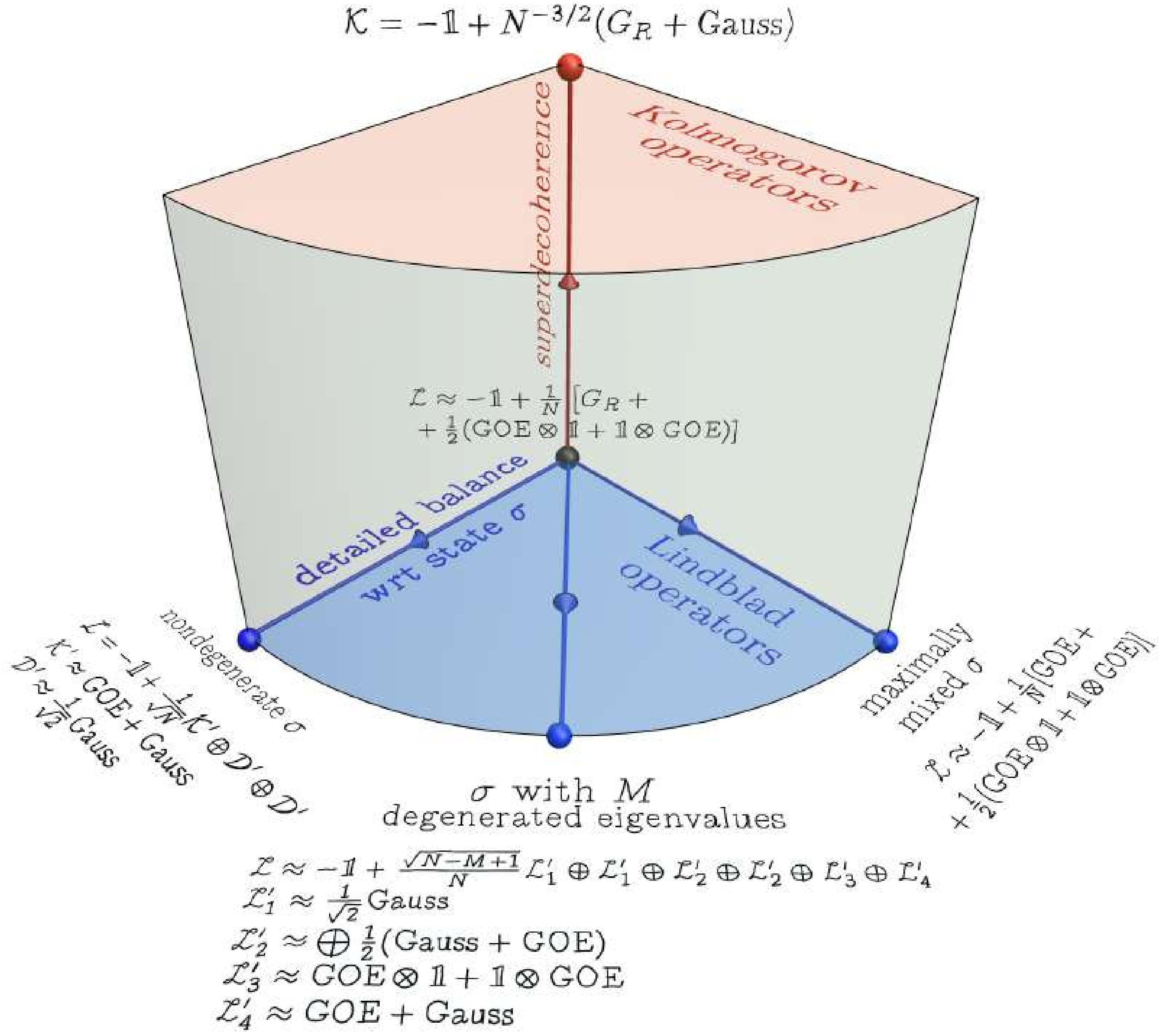
- 'Lemon RMM'

$$G_R - \frac{1}{2}(\mathrm{GOE} \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{GOE})$$

- detailed balance RMM (maximally mixed state)

$$\mathrm{GOE}_1 - \frac{1}{2}(\mathrm{GOE}_2 \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{GOE}_2)$$





W. Tarnowski, DC, S. Denisov, K. Życzkowski, OSID (2023)

Concluding remarks

- Spectra of random **quantum operation** consist of the leading eigenvalue $\lambda_0 = 1$ and the bulk of eigenvalues covering the Girko disk of radius $r = 1/N$, described by **real Ginibre ensemble**
- Spectra of random **Lindblad generators** consist of the leading eigenvalue $\ell_0 = 0$ and the bulk of eigenvalues covering the **lemon** shape with the spectral gap $\sim 1 - 2/N$
- Lemon-like boundary of the bulk was derived using free probability theory. The bulk can be described by free convolution of real Ginibre ensemble with the Hermitian correction term $\mathrm{GOE} \otimes 1 + 1 \otimes \mathrm{GOE}$.
- Universal features of Complex Spacing Ratio typical for GinUE (both classical and quantum)
- Variety of RMMs for detailed balance Liouvillians

Outlook

- no symmetries, full rank $\to$ lemon
- additional symmetry
- gradual superdecoherence from quantum to classical
- ...