

Full Quantum Work Statistics for Non-Homogeneous Many-Body Systems

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QOpen

Many-body Open Quantum Systems

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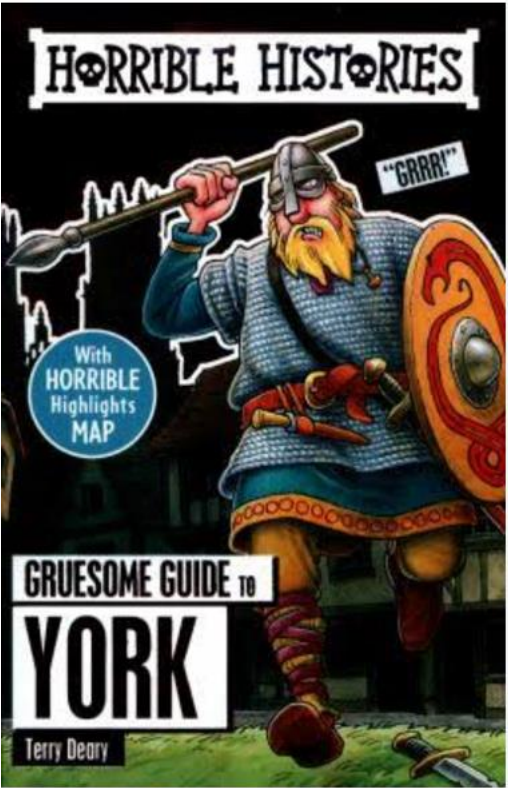


Antonello Sindona
University of Calabria

Pies and Ales!



Come and visit!



wikipedia

Ghosts and Hunted Pubs!



Vikings, Brrrrrr!

Motivations

- Emergence of irreversibility at the quantum scale
- Effects of quantum fluctuations on thermodynamic quantities/processes
- Related effects on *performance and energetics* of quantum (thermal) machines/technologies
- Improve methods to study many-body quantum systems (relevant to actual experimental set-ups and devices)

Motivations

- Improve methods to study many-body quantum systems (relevant to actual experimental set-ups and devices)
 - interactions
 - + non-homogeneous
 - + finite temperature
 - + non-equilibrium
 - + finite-time dynamics
 - +open system dynamics
- increased complexity
-  you must be joking!!

but: which quantities are we really interested in?
wave function/full system state? can we get away with less?

Today's focus:

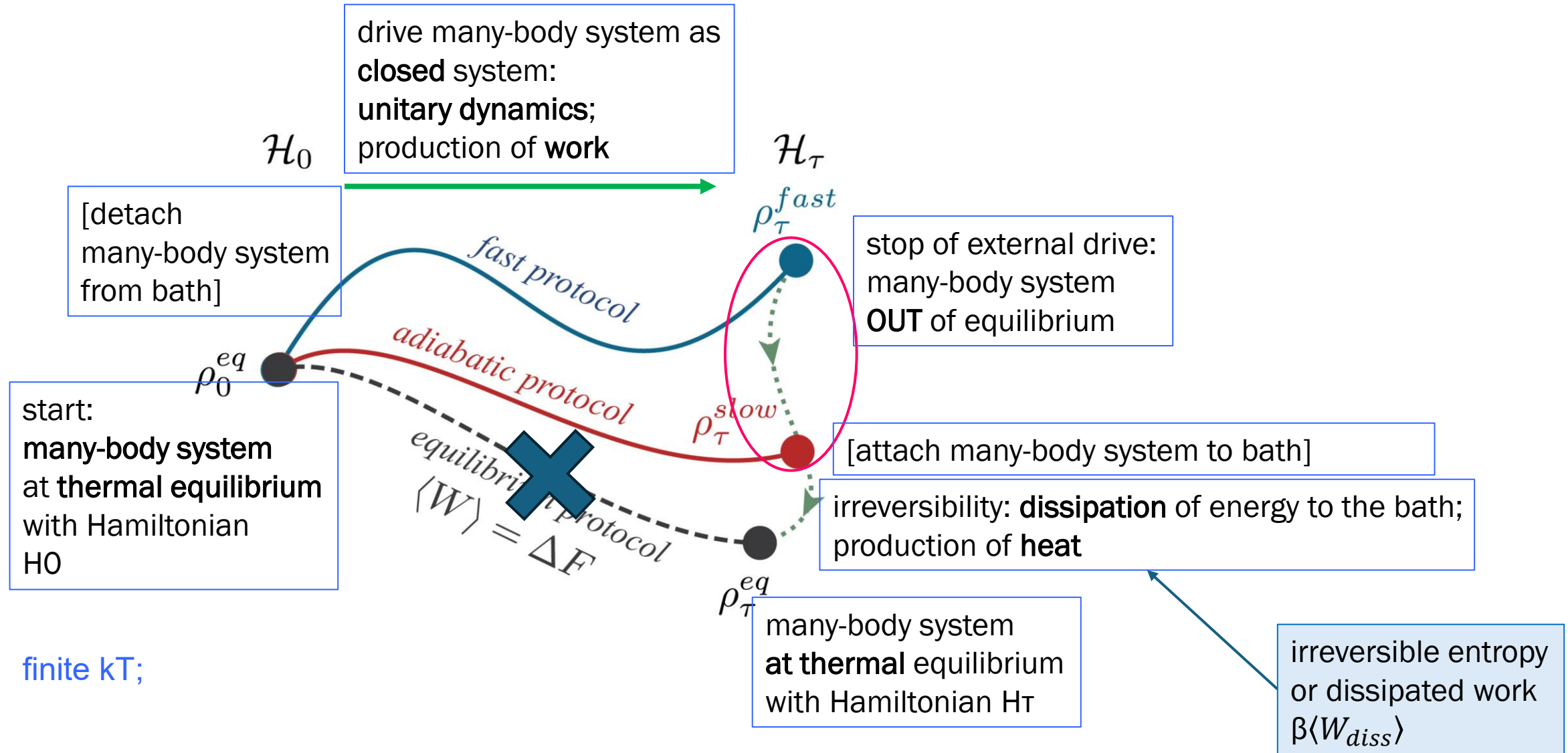
- Irreversibility and quantum fluctuations in the context of dissipative quantum work statistics
- Development of first-principle method for many-body quantum systems
 - interactions
 - + finite temperature
 - + non-equilibrium
 - + finite-time dynamics: [linear response](#)
- Method based on density functional theory (DFT)

Outline

- Linear response for dissipated-work probability distribution: connection to relaxation function
- Microscopic approach to the relaxation function:
 - Briefest introduction to Density Functional Theory (DFT) concepts
 - Some considerations on open quantum systems and DFT
 - Relaxation function from Linear-Response Thermal Time-dependent Density Functional Theory (LR-thTDDFT)
- Application to the ionic Hubbard model: phase diagram, quantum and thermal fluctuations
- Some conclusions & future work...

Linear response for dissipated- work probability distribution: connection to relaxation function

QTD protocol



Dissipated (or irreversible) work

S. Deffner and E. Lutz, PRL 107, 140404 (2011)

- measures the irreversibility of the thermodynamic process and is non-zero even for an adiabatic process, for close initially thermal systems:

$$\langle \Sigma \rangle = \beta(\langle W \rangle - \Delta F) = \beta \langle W_{\text{ir}} \rangle$$

- given by the relative entropy

$$\beta \langle W_{\text{diss}} \rangle = S(\hat{\rho}_\tau \parallel \hat{\rho}_\tau^{\text{th}})$$

- where

$$S(\rho_1 \parallel \rho_2) = \text{tr}\{\rho_1 \ln \rho_1 - \rho_1 \ln \rho_2\}$$

- defined via a two-point measurement scheme, it is a stochastic quantity with an associated probability distribution $P(W_{\text{diss}})$

Dissipative work and relaxation function

G. Guarnieri, J. Eisert, and H. J. D. Miller, Phys. Rev. Lett. **133**, 070405 (2024).

- Given the Hamiltonian

$$\hat{H}(t) = \hat{H}_0 + \lambda(t) \hat{V}$$

- with the system relaxation function

R. Kubo J Phys Soc Jpn 12, 570 (1957)

$$\psi(t) = \beta \int_0^\beta ds \langle \hat{V}(-is) \hat{V}(t) \rangle_0 - \beta^2 \langle \hat{V} \rangle_0^2 \quad \text{with} \quad \hat{V}(t) = e^{i\hat{H}_0 t} \hat{V} e^{-i\hat{H}_0 t}$$

- under linear response (small perturbation) cumulants of $P(W_{diss})$

$$\beta^n k_w^n = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \tilde{\psi}(\omega) \gamma_n(\omega) \left| \int_0^\tau dt \dot{\lambda}(t) e^{i\omega t} \right|^2 \quad \gamma_n(\omega) = \frac{1}{2} (\beta\omega)^{n-1} \begin{cases} \coth(\beta\omega/2), & \text{for even } n, \\ 1, & \text{for odd } n. \end{cases}$$

$$\beta \langle W_{diss} \rangle = \frac{1}{2} \int_0^\tau dt \int_0^\tau dt' \psi(t-t') \dot{\lambda}(t) \dot{\lambda}(t') \quad \text{for } n=1$$

Relaxation function: adiabatic and non-adiabatic

A. Palamara, F. Plastina, A. Sindona, Irene D'Amico arXiv:2512.18338
to be published in QST

- consider:

$$\psi(t) = \psi_{\text{NA}}(t) + \psi_{\text{AD}}$$

dynamical contributions arising from transitions between different energy levels

independent of the driving rate, reflects only the difference between the instantaneous spectra of initial and final Hamiltonians

Relaxation function: adiabatic and non-adiabatic

A. Palamara, F. Plastina, A. Sindona, Irene D'Amico arXiv:2512.18338
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ψ_{AD} :

- consider adiabatic process from $\lambda \rightarrow \lambda + \delta\lambda$ for $\tau \rightarrow \infty$
- calculate $\beta \langle W_{diss} \rangle = \beta \langle W_{diss} \rangle_{AD}$ from $\beta \langle W_{diss} \rangle = S(\hat{\rho}_\tau \| \hat{\rho}_\tau^{th})$
- expand to leading order in $\delta\lambda$ to obtain linear response expression

$$\beta \langle W_{diss} \rangle_{AD} = \frac{\delta\lambda^2 \beta^2}{2} \left(\sum_n p_n^0 |V_{nn}|^2 - \langle \hat{V} \rangle_0^2 \right)$$

- identify ψ_{AD} by comparing to corresponding linear response expression

$$\beta \langle W_{diss} \rangle = \frac{1}{2} \int_0^\tau dt \int_0^\tau dt' \psi(t-t') \dot{\lambda}(t) \dot{\lambda}(t')$$

- $\psi_{NA} = \psi(t) - \psi_{AD}$

Relaxation function: adiabatic and non-adiabatic

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- consider $\psi(t) = \psi_{\text{NA}}(t) + \psi_{\text{AD}}$

dynamical contributions arising from transitions between different energy levels

independent of the driving rate, reflects only the difference between the instantaneous spectra of initial and final Hamiltonians

- using perturbation theory up to second order in $\delta\lambda$ we obtain

$$\psi_{\text{NA}}(t) = \beta \int_0^\beta ds \langle \hat{V}(-is) \hat{V}(t) \rangle_0 - \beta^2 \sum_n p_n^0 |V_{nn}|^2, \quad \psi_{\text{AD}} = \beta^2 \left(\sum_n p_n^0 |V_{nn}|^2 - \langle \hat{V} \rangle_0^2 \right)$$

- from which

$$\tilde{\psi}(\omega) = \tilde{\psi}_{\text{NA}}(\omega) + \sqrt{2\pi} \delta(\omega) \psi_{\text{AD}}$$

time independent

also: Ma, Mitchell and Sela PRL 135, 130402 (2025)

Dissipated Work Distribution: adiabatic and non-adiabatic

A. Palamara, F. Plastina, A. Sindona, Irene D'Amico arXiv:2512.18338
to be published in QST

- consider

$$\beta^n k_{\text{w}}^n = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \tilde{\psi}(\omega) \gamma_n(\omega) \left| \int_0^{\tau} dt \dot{\lambda}(t) e^{i\omega t} \right|^2 \quad \gamma_n(\omega) = \frac{1}{2}(\beta\omega)^{n-1} \begin{cases} \coth(\beta\omega/2), & \text{for even } n, \\ 1, & \text{for odd } n. \end{cases}$$

- from $\tilde{\psi}(\omega) = \tilde{\psi}_{\text{NA}}(\omega) + \sqrt{2\pi} \delta(\omega) \psi_{\text{AD}}$
- we get adiabatic and non adiabatic contributions to $P(W_{\text{diss}})$
- from structure of $\gamma_n(\omega)$:
adiabatic components only for first two moments
 $P(W_{\text{diss}})$ is Gaussian in the adiabatic limit

- How to calculate the relaxation function from microscopic first principles?
- For a generic many-body interacting system?
- Let's go the Density Functional Theory way...

Briefest introduction to Density Functional Theory (DFT) concepts

Density Functional Theory (DFT)

- widely used for calculating energies of complex many-body systems and materials (' $T=0$ ')

Could help with

- many-body quantities (energy, entanglement, work...) from **local** properties (no need of wave function/system state)
- (good) approximation to many-body wave-function/system state
- expressions for non-local or 'non-observable' quantities in terms of local properties measurable in experiment

DFT: extensions and reach

?QOpen?

- ground state
- spin-dependent systems
- time-dependent systems (adiabatic regime) C. A. Ullrich's book on TDDFT (2012)
- lattice Hamiltonians (Hubbard, Heisenberg, XXZ...) Capelle and Campo Phys Rep 91, 528 (2013)
- thermal systems (in infancy) A. Pribram-Jones et al. (2014)
https://doi.org/10.1007/978-3-319-04912-0_2
- ensembles (in infancy) T. Gould et al. J. Chem. Phys. 164, 040901 (2026)
- open systems:
 - Markovian Lindblad equation K. Burke et al. PRL 94, 146803 (2005)
 - General master equation (including memory) J. Yuen-Zhou et al. PRL 104, 043001 (2010)

DFT: extensions and reach

- GS energy, band structure calculations; properties of materials (predictions and/or analysis from experiments); optical, opto-electronic and magnetic properties of nanomaterials; graphene...

RSC Adv., 2021, 11, 27897

- well established (public) codes: new functionals could be embedded in well-oiled machinery

- **ability of tackling very large many-body systems (even millions of atoms)**

J. Phys.: Condens. Matter **22** 074207 (2010);
Chemical Physics Letters 475 (2009) 163–170

DFT, all flavours based on:

- Hohenberg-Kohn-like theorems [Runge-Gross; Mermin;...]
- Kohn-Sham-type equations

Density Functional Theory: An Approach to the Quantum Many-Body Problem
Reiner M. Dreizler, Eberhard K. U. Gross, Springer 1990

A Bird's-Eye View of Density-Functional Theory
Klaus Capelle, Brazilian Journal of Physics, vol. 36, no. 4A (2006)

Time-Dependent Density-Functional Theory: Concepts and Applications
Carsten Ullrich, Oxford University Press (2012)

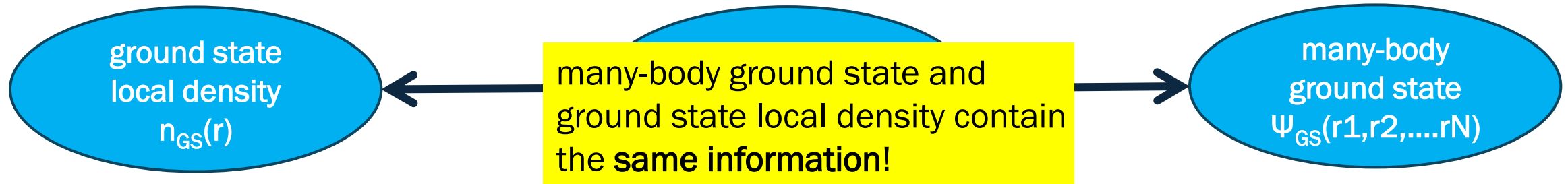
Density functionals and model Hamiltonians: Pillars of many-particle physics
K. Capelle and V. L. Campo Jr, Phys Rep 91, 528 (2013)

[.....] e.g. google DFT books or reviews

Hohenberg-Kohn (HK) theorem (Ground State, GS)

$$\hat{H}\Psi = \left[\hat{T} + \hat{V} + \hat{U} \right] \Psi = \left[\sum_{i=1}^N \left(-\frac{\hbar^2}{2m_i} \nabla_i^2 \right) + \sum_{i=1}^N V(\mathbf{r}_i) + \sum_{i<j}^N U(\mathbf{r}_i, \mathbf{r}_j) \right] \Psi = E\Psi$$

kinetic term
external potential
two-body interactions

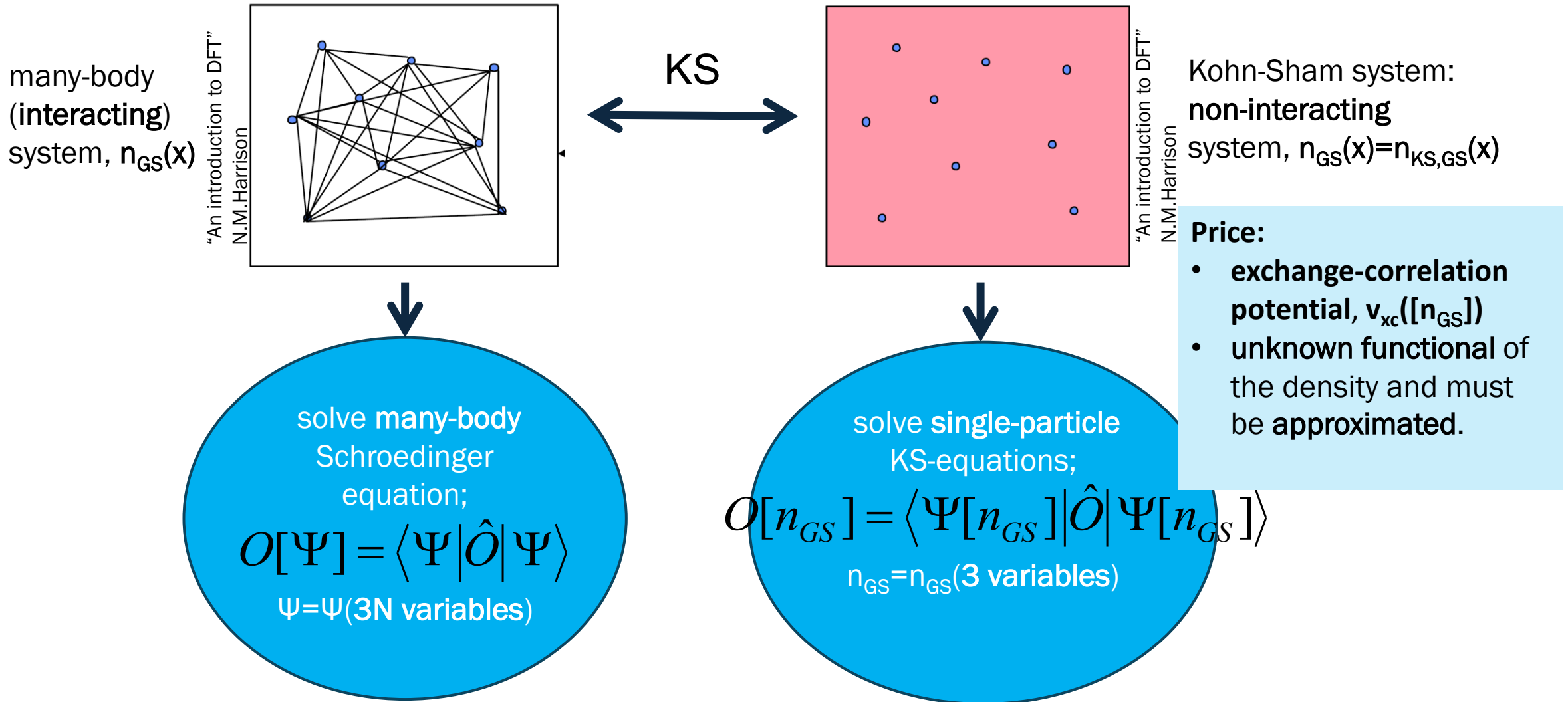


$$n(\mathbf{r}) = N \int d^3\mathbf{r}_2 \cdots \int d^3\mathbf{r}_N \Psi^*(\mathbf{r}, \mathbf{r}_2, \dots, \mathbf{r}_N) \Psi(\mathbf{r}, \mathbf{r}_2, \dots, \mathbf{r}_N)$$

implication: $O[\Psi] = \langle \Psi | \hat{O} | \Psi \rangle$ becomes: $O[n_{GS}] = \langle \Psi[n_{GS}] | \hat{O} | \Psi[n_{GS}] \rangle$

NB: existence of functional,
but not its form!

Kohn-Sham (KS) system/scheme:



Two different types of density functionals to be found or approximated:

1. **exchange-correlation potential, $v_{xc}([n])$:**
needed to solve the KS equations and find the local density n of the system
2. **functional dependence on the local density n of the quantity of interest,**
i.e. usually the explicit dependence on n of

$$O[n] = \langle \Psi[n] | \hat{O} | \Psi[n] \rangle$$

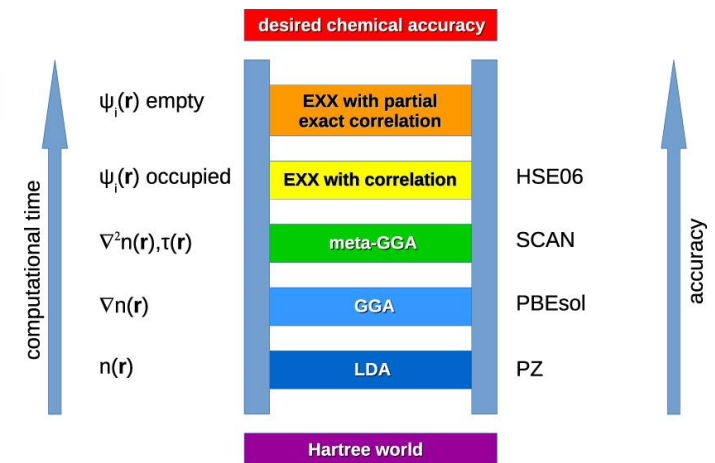
QTD: Two **plus one** different types of density functionals to be found or approximated:

1. exchange-correlation potential, $v_{xc}([n])$:
needed to solve the KS equations and find the local density n of the system
2. functional dependence on the local density n of the quantity of interest,
i.e. usually the explicit dependence on n of $O[n] = \langle \Psi[n] | \hat{O} | \Psi[n] \rangle$
3. **QTD:**
 - I. **demonstrate that non-local, non-observable quantities are functionals of the densities** (and which densities) e.g. entanglement & $P(W)$
 - II. **find the functionals** and their viable approximation

A Palamara, F Plastina, A. Sindona and I. D'Amico PRA 110, 062203 (2024)

The exchange-correlation potential v_{xc}

- The exchange-correlation energy E_{xc} (and its functional derivative v_{xc}) contains information about the many-body interactions of the original interacting system.
- In general v_{xc} is an **unknown functional** of the density and must be approximated (Jacob ladder).
- Most used: local density approximation (LDA).
LDA: at each point r , replace the exact xc energy density with that of the corresponding uniform, homogeneous system whose density has the same value as the local density $n(r)$.



Uncharted territory

DFT: functionals?
dependency on which local densities?
dependency of density functionals?
workable approximations for v_{xc} ?

- Demonstrate that QTD quantities are local-density functional
- Functionals of *which* local densities
- Find explicit expression for functionals
- Find workable approximations for functionals to apply KS scheme and/or time-dependent DFT
- Benchmark functionals and approximations to exact results
- (Find new results...)

$P(W)$, $P(W_{diss})$, sudden quench:

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Relaxation function from Linear-Response Thermal Time- dependent Density Functional Theory (LR-thTDDFT)

Relaxation function from Linear-Response

NA: Connection between relaxation function and response function (Kubo)

$$\frac{d\psi(t)}{dt} = \frac{d\psi_{\text{NA}}(t)}{dt} = -i\beta \sum_{ij} g_i g_j \langle [\hat{A}_i(t), \hat{A}_j(0)] \rangle_0. \quad \lambda_i(t) = [\lambda_0 + \delta\lambda(t)] g_i$$

and density-density response function $\chi_{ij}^{\beta}(t) = -i \theta(t) \langle [\hat{A}_i(t), \hat{A}_j(0)] \rangle_0$

gives
$$\tilde{\psi}_{\text{NA}}(\omega) = 2\beta \sum_{ij} g_i g_j \frac{\text{Im} \chi_{ij}^{\beta}(\omega)}{\omega}$$

$\text{Im} \chi_{ij}^{\beta}(\omega)$: excitations between different energy levels,
i.e. NA contributions

Relaxation function from Linear-Response

- AD: using expression for irreversible entropy from PRA 110, 062203 (2024)

$$\psi_{\text{AD}} = -\frac{\beta^2 \delta \lambda^2}{b(0)} \sum_{ij} g_i g_j \frac{\partial a_i^{\beta,0}}{\partial \lambda_j} - 2\beta \sum_{ij} g_i g_j \int \frac{d\omega}{\sqrt{2\pi}} \frac{\text{Im} \chi_{ij}^{\beta}(\omega)}{\omega} \frac{b(\omega)}{b(0)},$$

with $b(\omega) = \lim_{\tau \rightarrow 0} \left| \int_0^{\tau} dt \dot{\lambda}(t) e^{i\omega t} \right|^2$ and $\lambda_i(t) = [\lambda_0 + \delta \lambda(t)] g_i$

- now $\text{Im} \chi_{ij}^{\beta}(\omega)$ and $\frac{\partial a_i^{\beta,0}}{\partial \lambda_j}$ can be calculated with DFT methods

Equilibrium and KS quantities

- Many-body generalized time-dependent thermal densities $a_i^{\beta,0} = \text{Tr}\{\hat{\rho}_\beta^{\text{th}}[\{\lambda_i^0\}] \hat{A}_i\}$ can be obtained from the self-consistent non-interacting **KS Hamiltonian** $\hat{H}_{\text{KS}}[\{\lambda_{i,\text{KS}}(0)\}]$ with $\lambda_{i,\text{KS}}[\{a_i(0)\}]$ from which $\frac{\partial a_i^{\beta,0}}{\partial \lambda_j}$
- KS density-density response function $\chi_{\text{KS}}^\beta(\omega)$ can be obtain from KS equations (Lehmann representation)

Linear Response – th -Time Dependent DFT formalism (LR-thTDDFT)

- Under weak-enough time-dependent perturbation (linear response), the density response to changes is given by the density-density response function

$$\chi_{ij}^{\beta}(t) = -i \theta(t) \langle [\hat{A}_i(t), \hat{A}_j(0)] \rangle_0$$

- which can be obtained from the KS dynamics as

$$\chi^{\beta}(\omega) = \chi_{\text{KS}}^{\beta}(\omega) + \chi_{\text{KS}}^{\beta}(\omega) * \mathbf{f}_{\text{Hxc}}^{\beta}(\omega) * \chi^{\beta}(\omega)$$

- with Hxc kernel

LR-th: A. Pribram-Jones et al. PRL 116, 233001 (2016)

LR: Gross and Kohn PRL 55, 2850 (1985)

$$\mathbf{f}_{\text{Hxc}}^{\beta}(\omega) = [\chi_{\beta}(\omega)]^{-1} - [\chi_{\beta}^{\text{KS}}(\omega)]^{-1}$$

Recap of method

LR-thTDDFT provides the (in principle) exact relaxation function through the following steps:

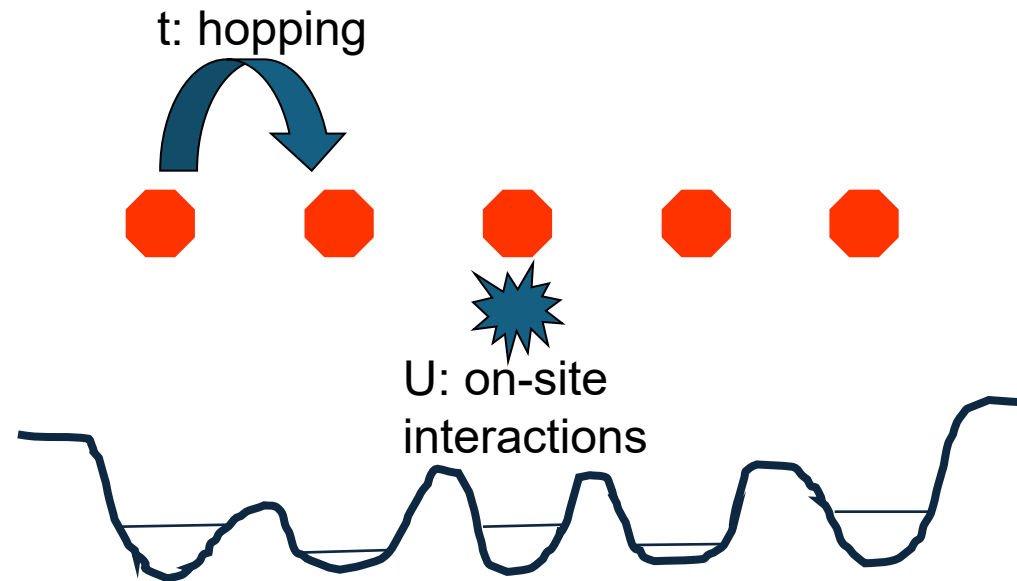
1. computation of the KS response function from thDFT;
2. inclusion of interaction effects via the Hxc kernel to obtain the fully interacting response;
3. extraction of $\text{Im } \chi_{ij}^{\beta}(\omega)$;
4. combination of the non-adiabatic contribution with the static adiabatic term to obtain the full relaxation function $\tilde{\psi}(\omega)$

Application to the ionic Hubbard model

1D Fermionic Hubbard model

- It allows for both itinerant electron spins (conduction band) and localized magnetic moments
- to describe ‘strongly correlated systems’; recently also cold atoms in an optical lattice, chains of trapped ions, excitons and electrons in coupled quantum dots, or small molecules
- half-filling, inhomogeneous chains with OBC

e.g. each site a quantum dot with a single energy level



each site 4 states:

$|0\rangle, |\uparrow\rangle, |\downarrow\rangle, |\uparrow\downarrow\rangle$

2 sites: 16 states

10 sites: 1,048,576 states

Driven 1D Fermionic Hubbard model

$$\hat{H}(t) = -J \sum_{i,\sigma}^N \overset{\text{hopping}}{\left(\hat{c}_{i,\sigma}^\dagger \hat{c}_{i+1,\sigma} + \hat{c}_{i+1,\sigma}^\dagger \hat{c}_{i,\sigma} \right)}$$

$$+ U \sum_i^N \hat{n}_{i,\uparrow} \hat{n}_{i,\downarrow} + \sum_i^N v_i(t) \hat{n}_i,$$

many-body interactions
external drive

with:

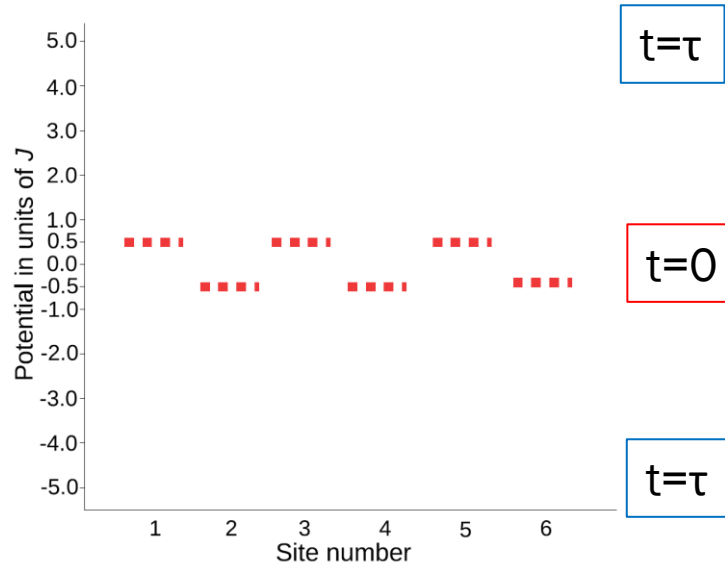
$$\hat{V}_{\text{EXT}}(t) = \sum_{i=1}^{\mathcal{L}} v_i(t) \hat{n}_i,$$

$$v_i(t) = v(t) (-1)^i$$

$$v(t) = v_0 + \frac{\delta v t}{\tau}$$

$$\delta v \ll v_0$$

$$t \in [0, \tau]$$



- half-filling,
- Open BC

Phase Transitions:

- fermions
- $U > 0$
- Metal to Mott Insulator (MIT) (driven by U)

- Transition to Band-Insulator (driven by **external drive**)

Bond-order insulator

thALDA approximation for Hxc kernel

$$f_{\text{Hxc}}^{\beta ij}[n_i^\beta] = \frac{\partial v_{\text{Hxc}}^\beta[n_i^\beta]}{\partial n_j^\beta} \delta_{ij}$$

with

$$v_{\text{Hxc}}^\beta[n_i^\beta] = U + \frac{1}{\beta} \ln \Gamma_U^\beta[n_i^\beta]$$

Single-site Hxc potential:

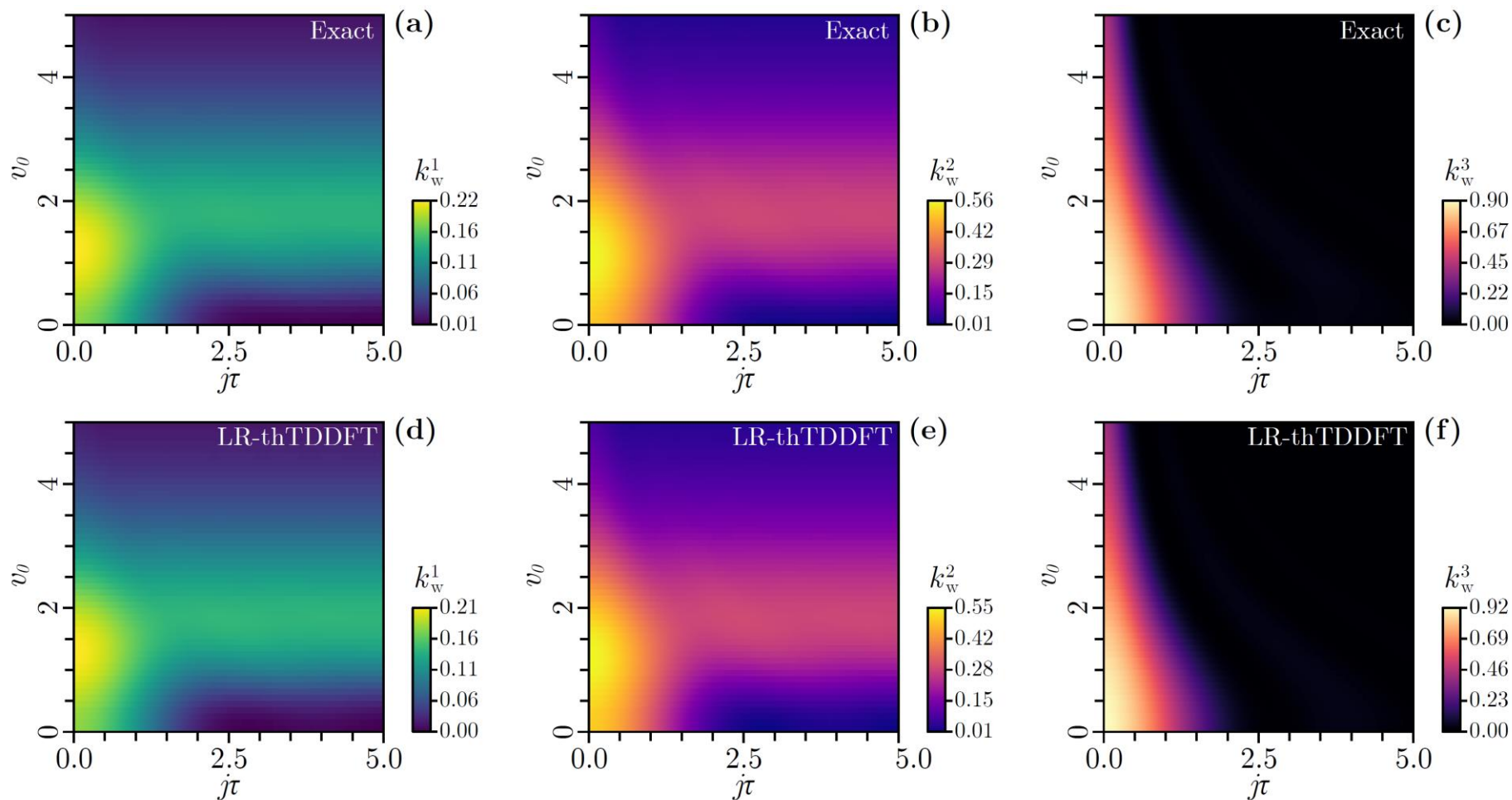
Hubbard chain is approximated to a collection of single sites coupled by the kinetic term

and

$$\Gamma_U^\beta[n_i^\beta] = \frac{(n_i^\beta - 1) + \sqrt{(n_i^\beta - 1)^2 + e^{-\beta U} [1 - (n_i^\beta - 1)^2]}}{n_i^\beta}$$

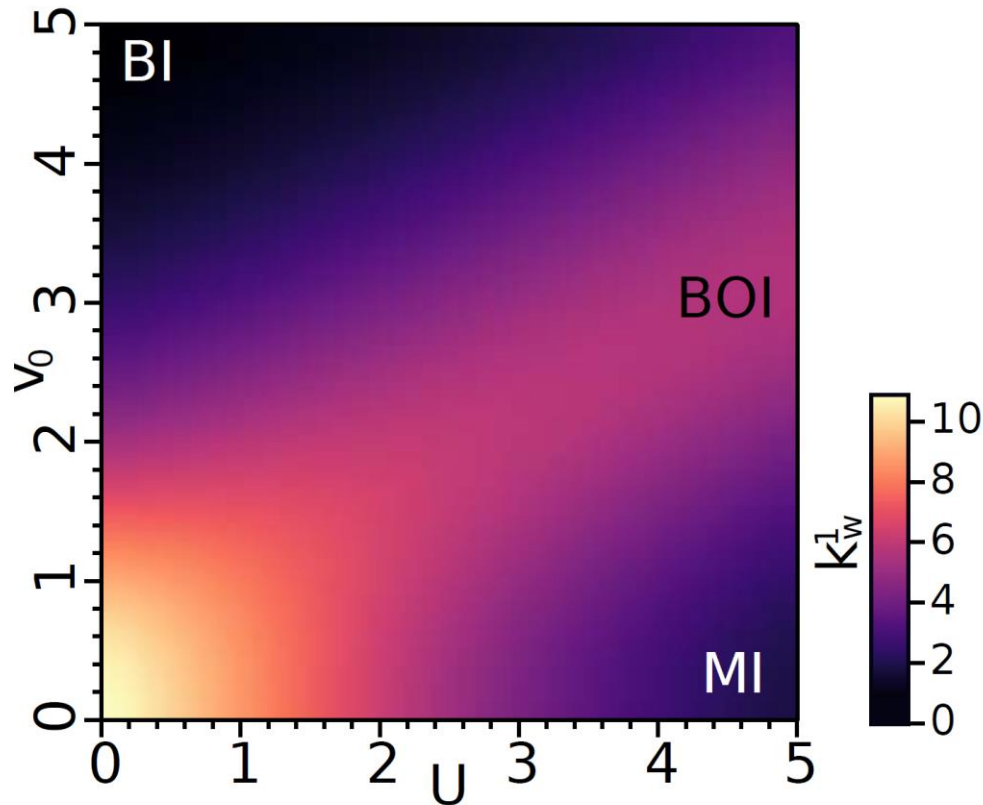
A Palamara, F Plastina, A. Sindona and I. D'Amico PRA 110, 062203 (2024)

Benchmarking: Exact vs LR-TDDFT, dimer



$U = 1 J$

Dissipated Work: Finite Temperature Phase Diagram with Staggered Applied Field



Dissipate work $\langle W_{diss} \rangle / (\delta\lambda)^2$
from sudden quench of the
control parameter

50 sites chain, half-filling

$\beta J = 1, \delta\lambda = 0.01J$.

BI : Band Insulator

BOI : Bond-order insulator [PRL 83, 2014 (1999)]

MI : Mott Insulator

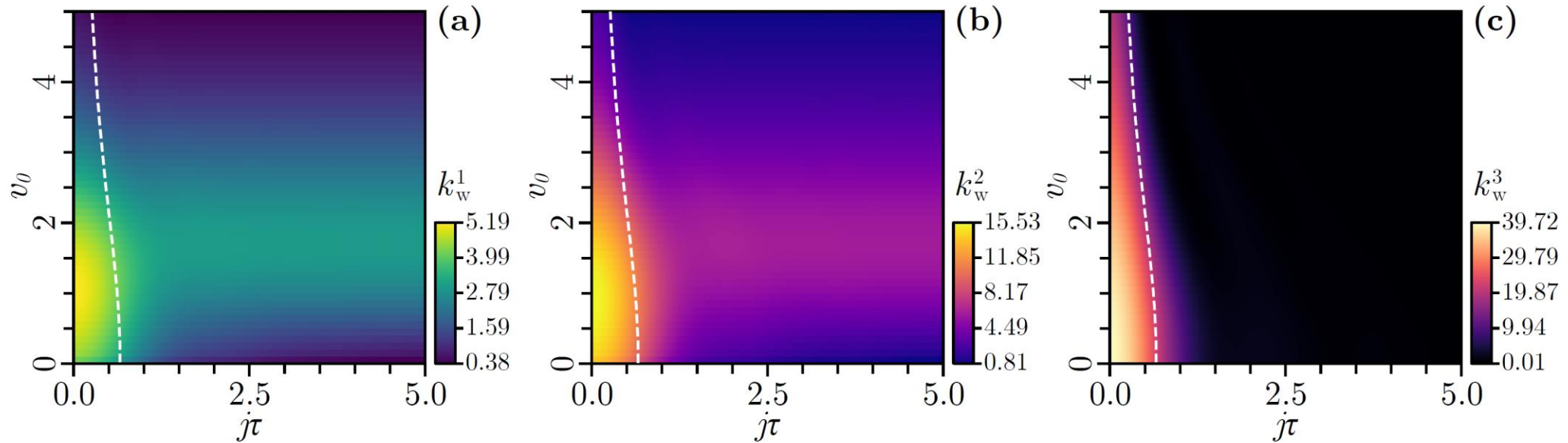
$$\beta k_w^1 = -\frac{\beta^2}{2} \sum_{ij} \delta\lambda_i \delta\lambda_j \frac{\partial a_i^{\beta,0}}{\partial \lambda_j^0}$$

$P(W)$, $P(W_{diss})$, sudden quench:

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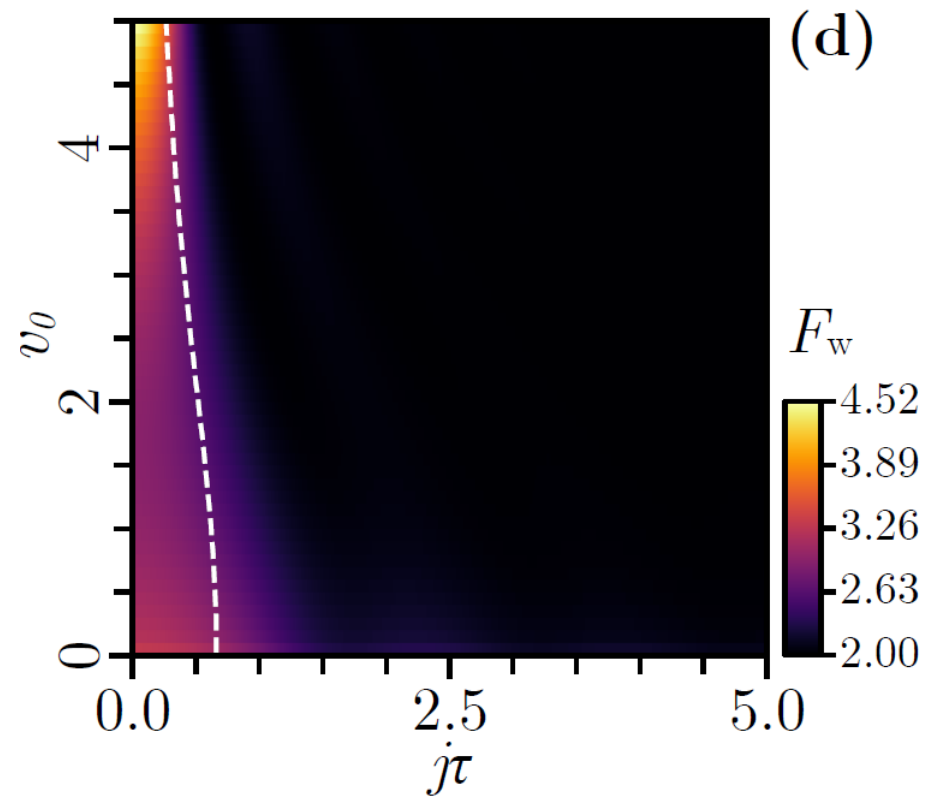
and I. D'Amico PRA 110, 062203 (2024)

Cumulants of the Work Distribution: Adiabatic vs Non-Adiabatic Regime



First 3 cumulants (central moments) of the work distribution vs quench duration τ .
50 sites chain, half-filling, $\beta J = 1$, $\delta\lambda = 0.01J$, $U = 1J$.
Dashed line: crossover from non-adiabatic to adiabatic regimes.

Fano factor



$$F_W = \frac{\text{Var}(W)}{\langle W_{\text{diss}} \rangle} \geq \frac{2}{\beta}$$

TUR is satisfied and saturated
only in the adiabatic regime:
importance of quantum fluctuations
especially in the non-adiabatic region

Conclusions

- Derived a microscopic approach to the relaxation function and hence to the full dissipated work statistics within Linear-Response Thermal Time-dependent Density Functional Theory (LR-thTDDFT) – AD and NA
- Provided a predictive approach applicable to a large class of many-body systems
- Application to the ionic Hubbard model: thermodynamic signatures of Mott, band and bond-order insulating phases
- Improvements: xc potentials / xc kernels beyond LDA /ALDA (dynamic correlations, memory)

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